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Disintegration Schemes of Radioactive Substances. VI. Mn^{56} and Co^{56}

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The radiations accompanying the decay of Mn^{56} and of Co^{56} were studied by means of the magnetic lens spectrometer and coincidence methods. Both of these nuclei decay to stable Fe^{56} as the product nucleus. Four excited states of the Fe^{56} nucleus are definitely established with excitation energies 0.845 Mev, 2.11 Mev, 2.66 Mev, and 2.98 Mev, respectively. De-excitation of the three high energy states always leads to the 0.845-Mev level and thence to the ground state. The negatron spectrum of Mn^{56} consists of three groups of maximum energies 2.86 Mev, 1.05 Mev, and 0.73 Mev, respectively, and relative abundance 60 : 25 : 15 leading to the first, third, and fourth of the excited states mentioned above. The positron spectrum of Co^{56} consists mainly of a single group of maximum energy 1.50 Mev, leaving the Fe^{56} nucleus in the 2.11-Mev state. Orbital electron capture also takes place involving several other excited states. Altogether the energies of eight gamma-rays emitted by the Fe^{56} nucleus are determined: 0.845 Mev, 1.26 Mev, 1.74 Mev, 1.81 Mev, 2.01 Mev, 2.13 Mev, 2.55 Mev, 3.25 Mev. Within the experimental uncertainty five of these energies appear to be the multiples 2, 3, 4, 5, and 6 of 0.425 Mev. All of the beta-ray spectra involved seem to have the "allowed" shape of the Fermi theory. The significance of this and of the relative probability of orbital electron capture is discussed in terms of the tensor interaction. It appears that all of the transitions should involve nuclear angular momentum changes $\Delta J = \pm 1$ or $\Delta J = 0$, with or without parity change. From the disintegration schemes one obtains the mass differences between neutral atoms, namely, $Mn^{56} - Fe^{56} = 3.98$ mMU and $Co^{56} - Fe^{56} = 4.96$ mMU, and the threshold for the reaction $Fe^{56}(p, n)Co^{56}$: 5.47 Mev.

INTRODUCTION

THE preceding papers of this series[1] have described investigations leading to the assignment of energy levels in product nuclei of radioactive disintegrations. The question naturally arises whether any particular radioactive decay involves all, or even most, of the energetically available levels of the product nucleus. The results of the previous investigations indicate that only a few of the possible transitions take place with sufficient frequency to be detected by our present methods. Thus, for example in the decay of I^{130} (paper V) only two of five possible beta-ray transitions and four of ten possible gamma-ray transitions between the known levels are actually observed. In the case of Br^{82} (paper II) only one of four possible beta-ray transitions and three of six gamma-ray transitions are found. So far it has always been possible to account for the absence of the other transitions on the basis of known selection rules for gamma-ray transitions and the Gamow-Teller selection rules for beta-ray emission. Consistent assignments of total angular momenta and parity have been possible, in this way, although these assignments are by no means unique. Generally the average level spacings are found to be of the same order of magnitude as

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[1] Phys. Rev. **60**, 470L (1941); **60**, 544 (1941); **61**, 686 (1942); **62**, 3 (1942); **64**, 268 (1943), hereafter referred to as papers I to V.

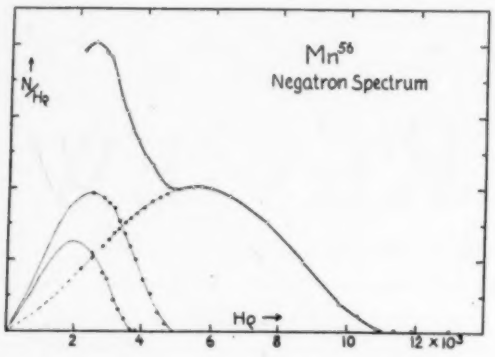

FIG. 1. The beta-ray spectrum of Mn^{56}. The open circles and heavy line represent the experimental data. The full circles indicate the three partial spectra, as derived from the Fermi plot.

those found in transmutation[2] and inelastic scattering[3] experiments. A comparison of the results of heavy particle studies with radioactive disintegration schemes should permit an estimate of the fraction of all states likely to be detected by any one of the methods. Unfortunately very few nuclei have been studied with sufficient accuracy by more than one of these methods giving information about low lying excited states.

It will probably be necessary to improve the resolution of the heavy particle studies, possibly by substituting magnetic analysis for the present absorption methods before they can be used for actual identification of energy levels. It is doubtful whether, e.g., the accuracy of any of the present transmutation and scattering experiments would be sufficient to resolve the two known excited states of Co^{59} (paper IV) or to decide the order of the two states in Ni^{60}.[4]

One may hope, however, to obtain valuable information by studying the radioactive decay of different parent nuclei to the same product nucleus. This may occur either by the decay of independent (genetically unrelated) isomers or by the decay of radioactive isobars differing by two in atomic number. The first case occurs rather rarely such as in Co^{60} which is, however, difficult to study[4,5] because of the short life of one of the isomeric activities. The convergent decay of two isobars occurs fairly frequently, but only a few of these isobaric pairs of activities are suitable for our purpose. In some cases (e.g., Cr^{52}) the half-life of one or both of the activities is inconveniently short. Others (e.g., Cu^{63}) lack interest because one or both of the disintegrations involves only the ground state of the product nucleus.

Preliminary investigations indicated that the isobaric pair $Mn^{56}-Co^{56}$ is very well suited for a study of the levels of Fe^{56} since both disintegrations involve several gamma-rays and proceed sufficiently slowly for careful investigation. A brief report on some results of such a study was made in a Letter to the Editor[6] and the present paper extends these results and describes the experiments leading to them.

A. Mn^{56}

The 2.59-hr. activity of manganese is definitely assigned to the species Mn^{56} since it is produced by slow neutron bombardment of stable Mn^{55}. The radiations accompanying its decay have been studied by a number of authors. We shall refer here only to some of the more recent results. Townsend[7] found that the beta-ray spectrum, observed in a 180° focusing spectrometer, appears to consist of two groups with maximum energies 2.88 Mev and 1.01 Mev, respectively. This is in good agreement with the results of Langer, Mitchell, and McDaniel[8] and of Dunworth[9] who

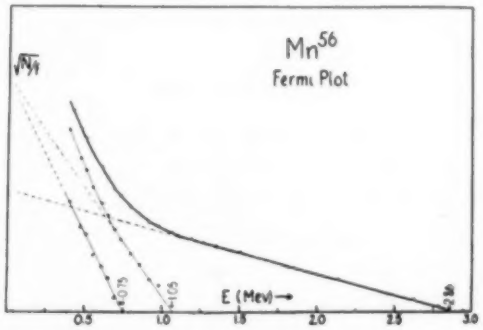

FIG. 2. Fermi plot of the spectrum shown in Fig. 1. The end points of the low energy spectra deduced from this plot are in excellent agreement with the level scheme found from gamma-ray energy and coincidence measurements.

[2] E. F. Shrader and E. Pollard, Phys. Rev. **59**, 277 (1941).
[3] R. H. Dicke and J. Marshall, Jr., Phys. Rev. **63**, 86 (1943).
[4] M. Deutsch and L. G. Elliott, Phys. Rev. **62**, 558A (1942).
[5] H. E. Nelson, M. L. Pool, and J. D. Kurbatov, Phys. Rev. **62**, 1 (1942).
[6] L. G. Elliott and M. Deutsch, Phys. Rev. **63**, 321L (1943).
[7] A. A. Townsend, Roy. Soc. Proc. **A177**, 362 (1941).
[8] L. M. Langer, A. C. G. Mitchell, and P. W. McDaniel, Phys. Rev. **56**, 422 (1939).
[9] J. V. Dunworth, Nature **143**, 1065 (1939).

found that all beta-rays of energy greater than about 1 Mev are accompanied, on the average, by the same gamma-rays whose energy the latter author estimated to be about 0.60 Mev, while the beta-rays of lower energy are accompanied by harder gamma-rays. Langer *et al.*[8] estimated the energy of these harder gamma-rays as 1.7 Mev. Gamma-gamma coincidences showed that some gamma-rays are emitted in cascade, and Dunworth[9] proposed a disintegration scheme in

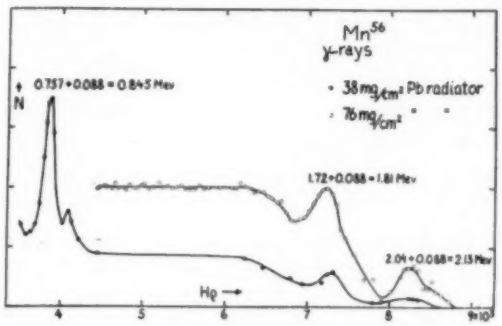

FIG. 3. Secondary electron spectra due to the gamma-rays of Mn^{56}. The two curves do not correspond to sources of equal strength.

which 60 percent of the beta-rays are accompanied by a single 0.6-Mev gamma-ray, while the remainder is accompanied by a 1.7-Mev and a 0.6-Mev gamma-ray in cascade. However, Curran, Dee, and Strothers[10] found two gamma-rays of 0.91-Mev and 2.01-Mev energy, respectively, by studying the Compton electrons ejected by the gamma-rays of Mn^{56} in a 180° focusing spectrometer. Deutsch and Roberts[11] studying photoelectrons as well as Compton recoils found values of 0.835 and 2.13 Mev, respectively. Although these measurements were not incompatible with Dunworth's disintegration scheme, the disagreement between the best beta-ray[7] and gamma-ray[11] energy determinations was serious enough to make the proposed scheme rather unsatisfactory. Recent improvements in spectrometer technique have allowed us to discover a third gamma-ray of energy 1.81 Mev which resolves these difficulties and leads to a satisfactory modification of the disintegration scheme.[6]

[10] S. C. Curran, P. I. Dee, and J. E. Strothers, Proc. Roy. Soc. **A174**, 546 (1940).
[11] M. Deutsch and A. Roberts, Phys. Rev. **60**, 362L (1941).

PREPARATION OF SOURCES

Sources of Mn^{56} were usually prepared by exposing a saturated solution of $KMnO_4$ to slow neutrons from the M.I.T. cyclotron for several hours. The solution was then filtered through a fritted glass filter, and most of the activity was found in the MnO_2 precipitate. Filtering the $KMnO_4$ solution through paper increases the amount of precipitate without increasing the activity recovered, probably because of the reducing action of the paper.

Thin sources for beta-ray studies were prepared by depositing a drop of active $MnCl_2$ solution and then a drop of $(NH_4)_2S$ solution on a thin flake of mica and evaporating to dryness. The source was then heated for several minutes to drive off excess reagent. This procedure is preferable to simple evaporation of a chloride solution since the finely divided sulfide precipitate forms a much more uniform and coherent deposit than a salt which crystallizes from the solution. The mica was mounted in the spectrometer without backing to minimize back scattering of the beta-rays.

For the study of gamma-ray energies, active MnO_2 was sealed into a brass capsule of sufficient wall thickness to stop all primary beta-rays. Suitable lead and bismuth radiators were fixed to this capsule with wax for the measurement of photoelectron distributions.

THE BETA-RAY SPECTRUM OF Mn^{56}

Figure 1 shows the momentum distribution of the beta-rays of Mn^{56}, obtained in the magnetic lens spectrometer. We believe that source and backing were thin enough to meet the rigorous criteria for reliable beta-ray measurements. Figure 2 shows the same data as a Fermi plot. Our results agree very closely with those obtained by Townsend.[7] It is quite obvious from Fig. 2 that there are at least two groups of beta-rays with maximum energies 2.86 Mev and about 1 Mev, respectively. The particular analysis of the Fermi plot into three straight lines as shown in Fig. 2 is based on evidence about the disintegration scheme obtained from the gamma-ray energies and coincidence measurements. It is by no means to be taken for granted that the Fermi plot can be analyzed into straight lines repre-

senting various groups since some simple "forbidden" spectra are known not to be represented by straight Fermi plots,[12] and it is unlikely that all of the groups emitted by Mn^{56} are emitted in "allowed" transitions. The significance of the fact that, nevertheless, when the beta-ray spectrum of Mn^{56} is analyzed into groups with the maximum energies predicted by independent information about the disintegration scheme, each group is represented by a straight Fermi plot, will be discussed further in Section C below. This result is similar to those obtained in the case of Fe^{59} (paper IV) and of I^{130} (paper V). The three groups are shown in Fig. 1. Their maximum energies are 0.75 ± 0.1 Mev, 1.05 ± 0.03 Mev, and 2.86 ± 0.05 Mev, respectively, and their relative abundance 15 : 25 : 60.

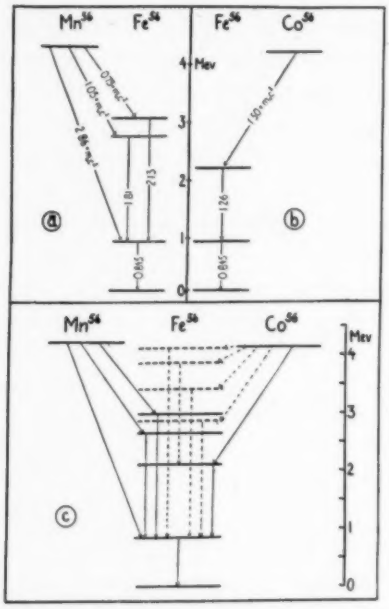

FIG. 4. a. Disintegration scheme of Mn^{56}. b. Level scheme for the most abundant mode of disintegration of Co^{56}. c. Proposed level scheme for the transitions to Fe^{56}. Solid lines and arrows indicate levels and transitions which appear well established. Dotted lines and arrows indicate an arrangement of the observed gamma-rays of Co^{56}, consistent with the probable selection rules.

THE GAMMA-RAY SPECTRUM

Figure 3 shows the spectra of secondary electrons produced in lead radiators of two different thicknesses (and in the brass capsule containing the source) by the gamma-rays of Mn^{56}. The observed counting rates were *not* divided by $H\rho$ as is customary, since the heights of the photoelectron lines, representing intensities of nearly monokinetic groups of electrons, should not be so divided.[13] There appear three pronounced photoelectron lines due to K electrons of lead ejected by gamma-rays of 0.845 ± 0.015-Mev, 1.81 ± 0.04-Mev, and 2.13 ± 0.04-Mev energy, respectively. The incompletely resolved L-electron group due to the lowest energy gamma-ray gives rise to the peak on the high energy side of the corresponding K electrons. The continuous background consists of Compton recoil electrons due to the same three gamma-rays. Only the energy range above 0.7 Mev is shown since no gamma-rays were found below that energy.

The resolution obtained in the experiments of Curran, Dee, and Strothers[10] and of Deutsch and Roberts[11] was not sufficient to resolve the two high energy gamma-rays.

The relative abundance of the three gamma-rays can be obtained from the heights of the photoelectron lines if the effect of finite radiator thickness and the variation of the photoelectric cross section with energy are considered. The result is:[14]

$$I_{2.13} : I_{1.81} : I_{0.845} = 20 : 30 : 100.$$

COINCIDENCE MEASUREMENTS OF Mn^{56}

Coincidences between beta- and gamma-rays were observed with varying amounts of absorber between source and beta-ray counter as well as coincidences between gamma-rays. The beta-ray counter used was of the helium-filled bell type described in paper II. The gamma-ray counters, also helium filled, had platinum screen cathodes (paper III). The arrangement of the counters, designed to reduce the number of spurious coincidences due to scattered radiations to a minimum, has been described in paper V. The results are in good qualitative agreement with those of other observers[8,9] The number of beta-

[12] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).

[13] L. G. Elliott, M. Deutsch, and A. Roberts, Phys. Rev. **63**, 386L (1943).

[14] Since the photoelectric cross section in this energy range is not well known experimentally, the theoretical values of Hulme *et al.*, Proc. Roy. Soc. **149A**, 131 (1935), were used.

gamma coincidences per recorded beta-ray is independent of beta-ray energy above about 1 Mev and has the value $1.00\pm0.05\times10^{-3}$ in the standard arrangement (paper V). When absorber is removed so that electrons of lower energy than 1 Mev are admitted into the beta-ray counter, the number of coincidences per recorded beta-ray increases rapidly and reaches a value of $1.92\pm0.08\times10^{-3}$ when all absorber except the counter window (5 mg/cm^2) is removed. The number of gamma-gamma coincidences per recorded gamma-ray is $0.92\pm0.07\times10^{-3}$ in the standard counter arrangement.

THE DISINTEGRATION SCHEME OF Mn^{56}

Since Mn^{56} can decay only by negatron emission, the level scheme shown in Fig. 4a seems to be the only one consistent with the experimental results. The fact that the high energy spectrum is accompanied by a single 0.845-Mev gamma-ray is established by the excellent agreement of the observed coincidence rate of 1.00×10^{-3} and the efficiency curve for the counter used (paper V). If either of the two high energy gamma-rays found in the secondary electron spectrum were emitted in transitions leading directly to the ground state, the maximum energy of the second beta-ray spectrum would have to be considerably higher than the value of about 1 Mev deduced both from the shape of the beta-ray spectrum and from the coincidence measurements. On the other hand there is not enough energy available for two high energy gamma-rays to be emitted successively. The proposed disintegration scheme predicts a value of 1.88×10^{-3} for the number of beta-gamma coincidences per beta-ray recorded without absorber, on the basis of the efficiency curve (Fig. 4, paper V) compared with the observed value of $1.92\pm0.08\times10^{-3}$. Similarly the predicted value of 0.99×10^{-3} gamma-gamma coincidences per recorded gamma-ray is in excellent agreement with the observed value of $0.92\pm0.07\times10^{-3}$. Thus the disintegration scheme of Mn^{56} may be considered well established.

B. Co^{56}

In contrast to Mn^{56} very little work has been done previously on the disintegration of Co^{56}. Livingood and Seaborg[15] have shown that the high energy positrons emitted with a half-life of 70–80 days by cobalt obtained from deuteron bombardment of either iron or nickel must be due to Co^{56}, since pure Co^{58} which was produced by alpha-particle bombardment of manganese emits much softer radiations. This was confirmed by producing Co^{58} by fast neutron bombardment of nickel.[16]

Cook and McDaniel,[17] using absorption and coincidence methods, reported that the positron spectrum is simple with a maximum energy of 1.2 Mev, that the average gamma-ray energy is 1.74 Mev, the energy of the highest energy gamma-rays 2.9 Mev, that the positrons are accompanied by gamma-rays, and that orbital electron capture seems to occur.

In a Letter to the Editor[6] we have described a partial disintegration scheme for Co^{56}. We shall now discuss the experiments leading to the adoption of that scheme and describe some other features of the disintegration of Co^{56}.

PREPARATION OF SOURCES OF Co^{56}

A study of the stable nuclear species in this region of the periodic table shows that it is impossible to produce Co^{56} free from other cobalt activities by any method used at present. It can be prepared by deuteron bombardment of

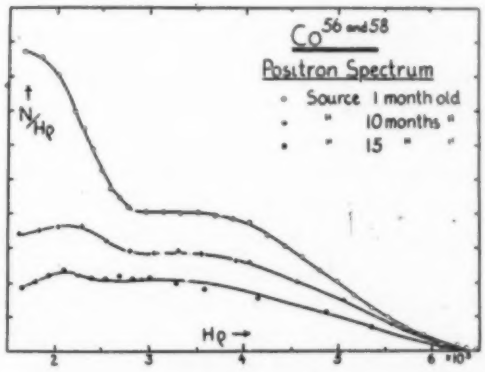

FIG. 5. Positron spectrum of radioactive cobalt separated from iron bombarded with deuterons. The curves represent spectra of three different sources.

[15] J. J. Livingood and G. T. Seaborg, Phys. Rev. **60**, 913L (1941).
[16] L. G. Elliott and M. Deutsch, Phys. Rev. **63**, 219A (1943). (See also paper VII, to appear soon.)
[17] C. S. Cook and P. W. McDaniel, Phys. Rev. **62**, 412 (1942).

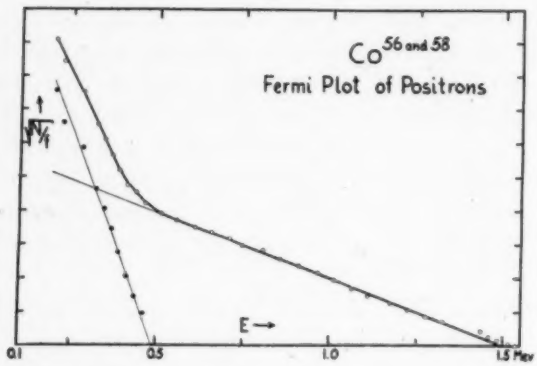

FIG. 6. Fermi plot of the positron spectrum of cobalt from (Fe+d). The plot represents the data shown in the top curve of Fig. 5. The low energy spectrum is due to Co^{58}.

iron in which case it is accompanied by Co^{55}, Co^{57}, and Co^{58}. Or it may be prepared by deuteron bombardment of nickel in which case Co^{58} and Co^{60} are also produced. We have studied sources prepared by both of these methods. Most of the experiments were carried out with samples obtained from iron. The iron targets were treated as described in paper IV. The filtrate of the pyridine precipitation was freed repeatedly of elements of the hydrogen sulfide group (by using inactive copper carrier) and cobalt was precipitated as potassium cobaltinitrite, and finally as oxide. Carrier material was added for all likely impurities, and all separations were repeated with additional carrier until all of the activity remained in the cobalt fraction.

Sources for beta-ray studies were prepared either by deposition as sulfide as in the case of manganese or by electroplating on thin brass or copper foil. Gamma-ray sources were prepared by sealing the oxide into a suitable brass capsule.

Samples prepared by deuteron bombardment of nickel were treated similarly.

THE BETA-RAY SPECTRUM OF Co^{56}

Figure 5 shows the positron spectra emitted by sources of cobalt prepared by deuteron bombardment of iron. These curves do not represent the decay of a single source but spectra of sources prepared at different times before the experiments. It is seen that the spectrum is complex and that the soft component decays somewhat more rapidly than the high energy radiation. Since the specific activity of the oldest source was rather low, the corresponding positron spectrum is probably distorted by scattering in the source material. An estimate of this decay showed that very few, if any, of the soft positrons of Co^{57} reported by Livingood and Seaborg[15] were present. On the other hand, the conversion lines due to the gamma-rays of Co^{57} with energies 0.119 and 0.131 Mev[18,19] were very pronounced, particularly in the older sources, and the intensity of these soft gamma-rays exceeded that of the gamma-rays of Co^{56} and Co^{58} many-fold. A discussion of Co^{57} will appear in another paper. Figure 6 shows the positron spectrum of the strongest source as a Fermi plot. The plot is a straight line above 0.48 ± 0.01 Mev and indicates a maximum energy of the positrons of 1.50 ± 0.05 Mev. After subtracting the high energy group, the low energy portion also appears to be represented by a straight Fermi plot. The maximum energy of 0.48 Mev suggests strongly that it is due to the positrons[16] of Co^{58}. This is supported by the more rapid decay of this group. Because of the presence of these positrons of Co^{58}, the low energy portion of the Co^{56} spectrum cannot be studied thoroughly, but it seems reasonably

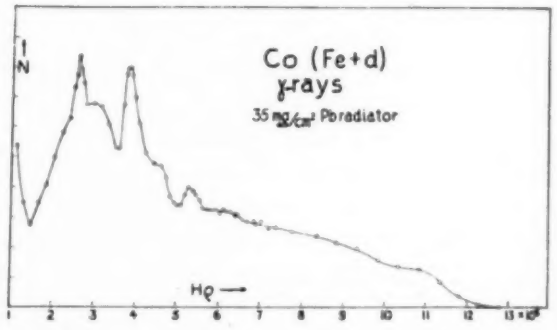

FIG. 7. Secondary electron spectrum due to gamma-rays of cobalt from (Fe+d), obtained with moderate resolution. The rapid rise near $H\rho = 1000$ is due to gamma-rays of Co^{57}.

certain that any low energy group of Co^{56}, if it exists at all, must be much less abundant than the main group of maximum energy 1.50 Mev. No decay curves were taken of any of the sources, but the variation of the relative intensity of the two spectra (and of the gamma-rays as discussed in the next section) in sources prepared at

[18] E. H. Plesset, Phys. Rev. **62**, 181 (1942).
[19] M. Deutsch, A. Roberts, and L. G. Elliott, Phys. Rev. **61**, 389A (1942).

different times from the same bombardment is consistent with the values of 85 days and 65 days for the periods of Co^{56} and Co^{58}, respectively.

THE GAMMA-RAY SPECTRUM OF Co^{56}

In a paper read before the Physical Society, Deutsch, Roberts, and Elliott[19] have reported gamma-ray energies of 0.119, 0.131, 0.510 (annihilation radiation), 0.835, 1.25, 3.40 Mev and a doubtful gamma-ray of 1.75 Mev from a mixture of long-lived cobalt activities produced by deuteron bombardment of iron. Figure 7 shows an example of the data on which that report was based. It was pointed out at that time that the photoelectron line due to the 0.835-Mev gamma-ray appeared wider than would be expected from instrumental effects and also that there was reason to believe that other gamma-rays exist of energies between 1.25 and 3.4 Mev. No trace of the 0.20- and 0.22-Mev gamma-rays reported by Plesset[18] was found, and it seems that the photoelectron lines ascribed by him to these gamma-rays were due to L and M electrons of lead ejected by the two known gamma-rays of lower energy definitely assigned to Co^{57} which were the only ones found with energy less than that of the annihilation radiation.

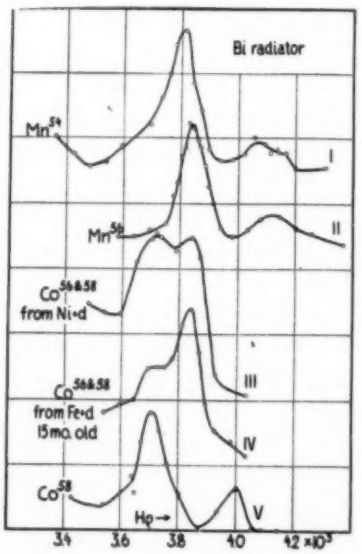

FIG. 8. Study of secondary electrons in the neighborhood of 0.75 Mev with high resolution. Comparison of curves II, III, and IV shows the identity of the energy of the 0.845-Mev gamma-rays of Mn^{56} and Co^{56}. Curves III, IV, and V show the varying relative abundance of Co^{58} in sources of mixed cobalt isotopes. Note separation of K and L photoelectron peaks in curves I, II, and V.

All secondary electrons of energy greater than 0.85 Mev decay at the same rate, as shown by the fact that the shape of the distribution curve was found to be independent of the age of the source. The relative abundance of the annihilation radiation and of the peak due to the 0.84-Mev radiation decreases with increasing age of the source while the intensity of the low energy radiation due to Co^{57} decays much more slowly than the rest of the radiations.

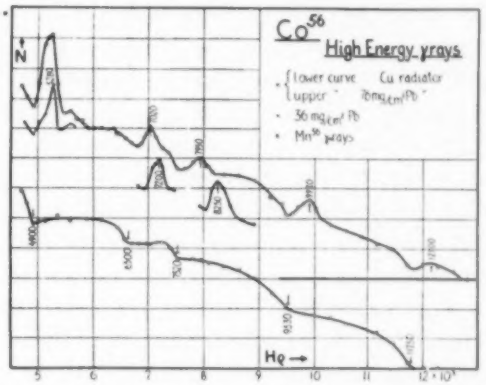

FIG. 9. Secondary electron spectra due to the high energy gamma-rays of Co^{56}, observed with good resolution. Numbers indicate the location of the peaks and high energy end points, respectively. Solid circles show the location of the peaks due to high energy gamma-rays of Mn^{56}, obtained with the same resolution.

Since a 0.805-Mev gamma-ray has been definitely assigned[16] to Co^{58} and since a 0.845-Mev gamma-ray was found in Mn^{56} which decays to the same product nucleus (Fe^{56}) as Co^{56}, the photoelectron group at 0.75 Mev, previously ascribed to a gamma-ray of 0.835 Mev, was investigated in more detail. The spectrometer was adjusted for better resolution and an arrangement was provided by which sources could be changed without disturbing the radiator. This eliminated any possibility of slight changes in the calibration of the spectrometer due to shifts in the effective source of photoelectrons. Figure 8 shows the result of this study. A thin bismuth radiator was used rather than lead to take advantage of the larger photoelectric cross section and to increase the separation between the K and L photoelectron groups. Curves I, II, and V of Fig. 8 show this separation and illustrate the resolution used. It will be seen that both of the sources containing mixtures of Co^{56} and Co^{58} show two peaks. One

TABLE I. Gamma-rays accompanying the decay of Co^{56}. Relative intensities referred to that of the 0.845-Mev gamma-ray. If all disintegrations involve this gamma-ray, relative intensities also represent number of quanta per disintegration. Note that intensity of annihilation radiation is only 0.5. If all transitions were by positron emission, this intensity should be 2.

E (Mev)	0.510 annihil. rad.	0.845	1.26	1.74	2.01	2.55	3.25
Intensity	0.5	1.0	0.5	0.2	0.1	0.2	0.2

of these corresponds accurately to the peak found with the source containing Co^{58} only. The other peak has an energy about 5 percent greater. The source prepared by deuteron bombardment of nickel shows a greater relative intensity of the Co^{58} gamma-ray, and a positron spectrum of the same sample showed a greater relative abundance of the Co^{58} spectrum. The higher energy peak obviously belongs to Co^{56}, and it is seen by a comparison of curves II and IV in Fig. 8 that its energy agrees very closely (within 0.5 percent or better) with that of 0.845-Mev gamma-ray of Mn^{56}.

Figure 9 shows a spectrum of secondary electrons due to the higher energy gamma-rays of Co^{56}. Because of the large number of gamma-rays in this region and their low relative intensities, it was necessary to take special precautions to reduce background effects (e.g., through coincidence counting) and to adjust the spectrometer for very good resolution. In this manner it was possible to resolve five distinct gamma-rays where our previous measurements[19] revealed only three. The upper curve shows the five groups of K photoelectrons, while the lower curve shows the Compton recoil groups due to the same five gamma-rays. The photoelectron group due to the highest energy gamma-ray appears somewhat wider than the others, suggesting the possibility of two unresolved peaks. Table I lists the energies of all of the gamma-rays of Co^{56} together with their estimated relative intensities.

Two of the gamma-ray energies, 1.74 Mev and 2.01 Mev, are within a few percent of the values 1.81 Mev and 2.13 Mev found in Mn^{56}. However, they are certainly different transitions. The peaks due to the Mn^{56} gamma-rays, obtained under the same geometrical conditions as those of Co^{56}, are also shown in Fig. 9. There can be no doubt but that the difference between the energies is real. It should be clearly understood that, while we estimate the probable errors in the absolute energy values to be about two percent, this allows for possible errors in the calibration of the instrument and the effect of finite radiator thickness, while relative energy determinations within a narrow energy range should be reliable to about 0.5 percent.

COINCIDENCE MEASUREMENTS OF Co^{56}

Coincidences between positrons and gamma-rays are studied preferably in the spectrometer since even with considerable precautions the annihilation quanta tend to introduce spurious effects when absorption methods are used. Coincidences between gamma-rays and positrons focused in the spectrometer were studied for several positron energies between 0.48 Mev and 1.50 Mev. About three millimeters of lead had to be inserted between the source and the gamma-ray counter in order to absorb the very intense gamma-radiation of Co^{57}. In this arrangement the number of coincidences per recorded positron was found to be independent of positron energy and to have the value $4.0\pm0.2\times10^{-3}$. By measuring coincidences between beta- and gamma-rays of Mn^{56}, Co^{58}, and Co^{60}, whose disintegration schemes are known, it was found that the coincidence rate expected if each positron of Co^{56} is accompanied by a 0.845- and a 1.26-Mev gamma-ray in cascade is $4.1\pm0.2\times10^{-3}$. This is in good agreement with the observed value. However, nearly the same coincidence rate would be expected if each positron were accompanied by a single gamma-ray of about 2-Mev energy. Therefore the average energy of the gamma-rays accompanying the positrons was determined by observing the variation of coincidence rate as lead absorber is placed between source and gamma-ray counter. Figure 10 shows the result of this experiment, corrected for chance coincidences. On the same figure is shown the absorption (in the same geometry) of the gamma-rays from Co^{60}, whose disintegration is known to be accompanied by two gamma-rays of 1.1- and 1.3-Mev energy, respectively.[4] The other dotted lines show the absorption expected with 2.1-Mev and with 0.85-Mev gamma-radiation. The absorption co-

efficients found in this geometry are slightly larger than the accepted values because of the obliquity of the gamma-rays traversing the absorber. It appears that the observed absorption for the case of Co^{56} agrees very well with that expected for a 0.845-Mev and a 1.26-Mev gamma-ray in cascade, but would be quite inconsistent with the assumption of a 2-Mev gamma-ray.

X-ray-gamma and gamma-gamma coincidences were also observed, but because of the unavoidable presence of other cobalt activities, particularly Co^{58}, the results have little quantitative significance.

THE DISINTEGRATION SCHEME OF Co^{56}

In the preceding section it was shown that the positron spectrum of Co^{56} is accompanied by a 1.26- and a 0.845-Mev gamma-ray as shown in Fig. 4b. The order of emission of the two gamma-rays is not directly established by experiment but the arrangement chosen in Fig. 4b seems very probable since a state of 0.845-Mev excitation energy in the Fe^{56} nucleus was previously found in the disintegration of Mn^{56} (Fig. 4a).

The positron spectrum is certainly simple above 0.48 Mev and if any branching occurs at lower energies, the low energy spectrum is almost certainly not very abundant because of the competing orbital electron capture and the relatively low intensity of the high energy gamma-rays. It appears that the high energy gamma-rays accompany orbital electron capture almost exclusively.

Any attempt to arrange these high energy gamma-rays into a level scheme is necessarily largely conjectural at the present time. A few features of the scheme can, however, be established with a certain degree of probability: It seems very unlikely that either the 1.74-Mev or the 2.01-Mev gamma-ray is emitted in a transition directly to the ground state from a level to which orbital electron capture takes place, since positron transitions to such a level would almost certainly be observed. The same argument holds—with less force—for the 2.55-Mev gamma-ray and for transitions of a 1.74-Mev gamma-ray to the 0.845-Mev level. The fact that the 0.845-Mev gamma-ray is about twice as abundant as the 1.26-Mev gamma-ray must have one of three possible explanations: 1. Orbital electron capture directly to the 0.845-Mev state. 2. Other transitions of the same energy but between two excited states. 3. Other gamma-ray transitions, besides the 1.26-Mev gamma-ray, leading to the 0.845-Mev level.

The first possibility seems extremely unlikely. In view of the great energy available for this transition (3.3 Mev), positron emission would certainly compete to a considerable extent with electron capture; and positron emission leading directly to this level is not observed. The second cannot be entirely excluded but seems somewhat improbable. The most reasonable assumption is the third. In Fig. 4c all of the states of Fe^{56} are definitely established and the transitions between them are indicated by solid lines, while the dotted lines indicate an assignment of the remaining gamma-rays to terms which are consistent with the above considerations. This particular arrangement is rather arbitrary and must necessarily remain so until further improvements in technique permit, for example, the study of coincidences between selected gamma-rays.

The fact that the annihilation radiation is only half as abundant as the 0.845-Mev gamma-ray indicates that the total probability of orbital electron capture to all states is at least three

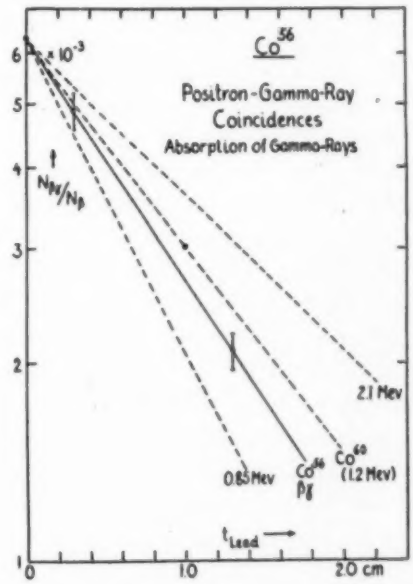

FIG. 10. Absorption of the gamma-rays coinciding with the positrons of Co^{56}.

times as great as that of positron emission (except on assumption 2 above). The relative probabilities of the two modes of transition to the 2.11-Mev level alone cannot be determined, however, since part of the disintegration scheme is not yet fully understood. For the particular scheme indicated by dotted lines in Fig. 4c the probability of orbital electron capture leading to the 2.11-Mev level is about one-fifth of that of positron emission.

C. DISCUSSION

The existence of four excited states in Fe^{56} with excitation energies of 0.845 ± 0.015 Mev, 2.11 ± 0.04 Mev, 2.66 ± 0.05 Mev, 2.98 ± 0.06 Mev has been definitely demonstrated. Of these levels only the lowest one is excited both by the decay of Mn^{56} and of Co^{56}. It can be seen, then, that neither disintegration involves *all* of the excited states which are energetically possible, and it is reasonable to assume that there are other states which are not excited in either disintegration. On the other hand the two sets of levels are not entirely unrelated since they have the lowest excited state in common.

Inspection of the eight gamma-ray energies assigned to the Fe^{56} nucleus shows that there are several cases in which two of them may be added to obtain the energy of a third observed gamma-ray, or where one gamma-ray has just twice the energy of another. It has been shown in other disintegrations, e.g.,[13] Na^{24} and Br^{82} (paper II) that such energy relationships do not necessarily mean that the same states are involved in the transitions. In Fe^{56}, too, we find, for example, that the excitation energy of the state to which positron emission takes place agrees, within the probable error, with the energy of the 2.13-Mev gamma-ray emitted following negatron emission by Mn^{56}.

Guggenheimer,[20] extending a suggestion of Rutherford and Ellis,[21] has pointed out that this behavior may not be fortuitous, since the energies of the gamma-rays emitted by some radioactive substances seem to occur in simple ratios. In Table II we list all of the gamma-rays emitted by the Fe^{56} nucleus. Inspection of the column headed "$E/0.425$" shows that five of the eight gamma-ray energies are integer multiples of 0.425 Mev, within the experimental uncertainty. It is particularly noteworthy that all integers from two to six are represented. The column headed "Terms" shows the energies of the initial and final states of the transitions wherever these are known.

The fact that the Fermi plots of the various beta-ray spectra are all straight lines within the experimental accuracy is rather surprising, since the positron spectrum of Co^{56} and probably also the high energy negatron spectrum of Mn^{56} must be emitted in "forbidden transitions," if this expression retains any significance with respect to a "Sargent diagram."

This means that all of the transition must involve differences of 0 or 1 in nuclear angular momentum between initial and final state (in the tensor theory), with or without parity change, the dominant matrix element being $\int\alpha$ or $\int 1$, respectively, in the notation of Konopinski and Uhlenbeck.[12] The shape of the lowest (and possibly also the intermediate) spectrum of Mn^{56} is somewhat uncertain, though. It is interesting to note that the ratio of the probability of orbital electron capture to that of positron emission predicted by theory[22] for an allowed transition in the case of Co^{56} is $\lambda_k/\lambda_{e^+}=0.25$ in close agreement with the value derived from the scheme shown in Fig. 4c. If the transition is first forbidden, this value could still be consistent with the theory for the tensor interaction in certain transitions if the dominant matrix element is $\int\alpha$.[12] If the transition is of a different type, agreement with theory could be obtained only by revising the uncertain parts of the level scheme.

TABLE II. Gamma-rays emitted by the nucleus Fe^{56}.

E	Disintegration	Terms	$E/0.425$
0.845	Mn^{56}, Co^{56}	0.845–0	1.99 (2.0)
1.26	Co^{56}	2.11 –0.845	2.97 (3.0)
1.74	Co^{56}	?	4.09 (4.0)
1.81	Mn^{56}	2.66 –0.845	4.26 —
2.01	Co^{56}	?	4.73 —
2.13	Mn^{56}	2.98 –0.845	5.01 (5.0)
2.55	Co^{56}	?	6.00 (6.0)
3.25	Co^{56}	?	7.65 —

[20] K. H. Guggenheimer, Proc. Roy. Soc. **A181**, 169 (1942).

[21] Lord Rutherford and C. D. Ellis, Proc. Roy. Soc. **A132**, 667 (1931).

[22] R. E. Marshak, Phys. Rev. **61**, 431 (1942).

TABLE III. Possible angular momentum quantum numbers and parity of states of Mn, Fe, and Co.

Nucleus	Excitation energy Mev	Angular momentum $h/2\pi$	Parity
Fe^{56}	0	0	*e*
	0.845	1	*o*
	2.11	6	*e*
	2.66	3	*o*
	2.98	3	*o*
Mn^{56}	(4.21)	2	*o*
Co^{56}	(4.12)	6	*o*

It is possible to explain the absence of cross-over transitions in the disintegration scheme (Fig. 4c) on the basis of the known selection rules for beta- and gamma-ray transitions, as was done in other papers of this series. Table III shows a set of such angular momentum and parity assignments (the parity of a state is indicated by *e* if it is even, by *o* if it is odd). Of course, all the parities could be reversed without changing the character of the transition. The reasoning leading to these assignments is similar to that in paper V. The values proposed are not particularly convincing, and they are given only to show that no further selection rules are required to explain the present data.

From the total disintegration energies of the schemes in Figs. 4a and 4b we can calculate the mass differences between the neutral atoms involved. We find $Mn^{56}-Fe^{56}=3.98\pm0.05\times10^{-3}$ aMU $Co^{56}-Fe^{56}=4.96\pm0.06\times10^{-3}$ aMU.

Similarly one can calculate the threshold for the reaction $Fe^{56}(p, n)Co^{56}$ from the disintegration energies, taking into account the recoil energy of the product nucleus and the mass difference between neutron and proton ($n-p=1.26\pm0.02\times10^{-3}$ aMU). The threshold energy is thus found to be 5.47 ± 0.12 Mev. This high value makes Fe^{56} an interesting nucleus for the study of inelastic proton scattering, since it allows the use of high proton energies for the formation of excited states up to 5 Mev without competition from neutron emission. The much lower threshold[16] for the reaction $Fe^{58}(p, n)Co^{58}$ should eliminate interference by protons scattered inelastically by Fe^{58}.

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On Applications of the λ-Limiting Process to the Theory of the Meson Field

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The λ-limiting process introduced by Wentzel and Dirac for the interaction of an electron with an electromagnetic field is applied to the interaction of a heavy particle (nucleon) with the meson field. One obtains in this way a more precise and relativistically invariant form of the theory which, in its classical interpretation, avoids all divergences of point sources. It is shown that the pseudoscalar and vector theories with a coupling constant f of the dimension of length have in this form the character of a weak coupling theory with no stable isobars existing for $\mu f \ll 1$. The higher approximations for the interactions between nucleons are investigated for the example of a Rosenfeld-Møller mixture with the results that they are finite as long as the nucleons are treated to be at rest and small in comparison with the f^2 approximation for distances of the nucleons from each other sufficiently larger than f.

1. INTRODUCTION

THE previous form of the theory of interaction of a meson field with heavy particles (protons and neutrons, which are both called also "nucleons") suffered from the deficiency of a fundamental character in that the range of validity of the perturbation theory with respect to the coupling constant of this interaction could not be stated without introducing new assumptions about the size of the nucleons. Since the higher approximations are divergent for point sources, one tried first to assume extended sources with a certain finite radius a. It turned out, however, that for the pseudoscalar and vector theories which alone give the right spin dependence of the nuclear forces, the perturbation theory never holds for values of a and of the coupling constant f with the dimension of length[1] which are required by experience. On the contrary, for sources small in comparison with the Compton wave-length μ^{-1} of the meson and for $a \ll f$, the different approximation of strong coupling has to be applied. These strong coupling theories, however, had consequences in contradiction to experiment.[2] Because of the existence of stable isobars with higher values of spin and charge, the highly charged nuclei should be unstable; moreover, the magnetic moment of the proton and the free neutron should be just equal apart from their sign and should nearly cancel in the deuteron in contradiction to the experimental facts.

One had the impression that the results of the perturbation theory are actually valid in a wider range than would follow from the model of the extended source. In the present paper[3] it is shown that this is understandable and can be formulated more precisely with the help of the so-called "λ-limiting process" which was applied in the case of the electromagnetic field by Wentzel and Dirac in order to avoid the classical singularities of the self-energy of a point electron[4] in a relativistically invariant way. In Section 2 it is shown that this limiting process can be considered as the result of a natural generalization of the model of an extended source with respect to the reality conditions of the field variables in the momentum space, and that the main physical content of this formalism, which has the further advantage of preserving the relativistic invariance of the final results,[5] can be characterized as making the constants of inertia of the degrees of freedom of the nucleon's spin and the isotopic spin equal to zero. These latter constants were

[1] In the following we use the natural unit where one puts $\hbar = c = 1$.

[2] W. Pauli and S. Kusaka, Phys. Rev. **63**, 400 (1943), where other literature is referred to.

[3] A short abstract of the contents of this paper was given in Phys. Rev. **63**, 221A (1943), No. 25.

[4] Compare the literature quoted in the author's report in Rev. Mod. Phys. **15**, 175 (1943).

[5] In the present paper we do not give an explicit proof for this relativistic invariance, which can be done for the Hamiltonians of the type used in Section 3 with the help of Dirac's formalism with several time coordinates in a way similar to Dirac's proof for the case of electrodynamics. Compare also J. M. Jauch, Phys. Rev. **63**, 334 (1943) where the application of the λ-limiting process to the problem of the magnetic moment of the nucleons is treated.

defined by Bhabha[6] in his classical theory of neutral vector mesons, and he proposed also to assume their values equal to zero for the actual particles of nature.[7] The discussion of the classical model shows that no stable excited states of the nucleons exist in this theory if $\mu f \ll 1$, a condition which certainly is fulfilled in nature and which characterizes the theory here discussed as a weak coupling theory in a more precise form.

The Hamiltonians for the interaction between nucleons and meson field given in Section 4 do not contain anything new and are only preparation for the investigation of the interaction between different nucleons in higher approximations treated in the concluding Section 5. We use here the important method of successive canonical transformations which is due to Stueckelberg and Patry,[8] and we are particularly interested in the special mixture of a pseudoscalar and a vector meson field introduced by Rosenfeld and Møller[9] which seems to be in better agreement with the empirical form of the nuclear forces than other assumptions about the meson field. While Rosenfeld and Møller themselves merely guessed that the higher approximations for the nuclear forces would be relatively small if the distance r between the nucleons is larger than the coupling constant f mentioned above, Stueckelberg[10] showed that the higher approximations diverge for all distances r if the model of a point source is used because these higher approximations contain terms proportional to a^{-1}, where a is the radius of the nucleon. This result is in agreement with the later investigations from the standpoint of the strong coupling[11] theories in which the stable isobars of the nucleon play an essential role.

In Section 4 we resume Stueckelberg's investigation of the Rosenfeld-Møller mixture in order to apply the λ-limiting process to this problem. For this purpose the use of the momentum space instead of the ordinary space in the description of the field variables is convenient, and it also simplifies the calculation itself. The result is the re-establishment of Rosenfeld and Møller's original condition $r \gg f$ for the validity of the perturbation theory at least as far as the order of magnitude is concerned. It is true that the numerical factors in the higher approximations tend to diminish this range of validity of the f^2 approximation, but it has to be remembered that the other point of view, which is used in the above calculations and according to which the nucleons are considered to be at rest, restricts the validity of the results to $r \gg M^{-1}$, where M^{-1} is the Compton wave-length of the proton, which is practically of the same order of magnitude as f. It is therefore questionable whether the exact form of the higher approximations with respect to f^2 has any significance, and we merely stress their order of magnitude and the circumstance that the λ-limiting process is sufficient to make them all convergent so long as the nucleons can be assumed to stay at rest.

2. THE λ-LIMITING PROCESS AS A GENERALIZATION OF THE EXTENDED SOURCE MODEL

Choosing as the simplest example the interaction of neutral pseudoscalar mesons with a heavy particle (nucleon) at rest in the origin of the coordinate system, we start with the well-known Hamiltonian

$$H = \tfrac{1}{2}\int \{\pi^2 + (\nabla\varphi)^2 + \mu^2\varphi^2\} d\mathbf{x} + (4\pi)^{\frac{1}{2}} f \int U(x)\boldsymbol{\sigma}\cdot\nabla\varphi d\mathbf{x}, \quad (1)$$

where the real quantities $\pi(x)$, $\varphi(x)$ are canonically conjugate pseudoscalar fields; natural units with $\hbar = c = 1$ are used, μ is the rest mass of the meson in these units, f the coupling constant with the dimension of length, $\frac{1}{2}\boldsymbol{\sigma}$ the spin of the nucleon, and $U(x)$ the source function normalized according to

$$\int U(x) dV = 1. \quad (2)$$

If brackets mean the Poisson symbols in the classical interpretation, and the commutators

[6] H. J. Bhabha, Proc. Roy. Soc. **A178**, 314 (1941).

[7] It is an interesting question whether the λ-limiting process can be generalized relativistically invariant in such a way that this constant is given an arbitrary value different from zero and that divergences of theory are still avoided. My attempts to find such a generalization have not been successful.

[8] E. C. G. Stueckelberg and J. F. C. Patry, Helv. Phys. Acta **13**, 167 (1940).

[9] C. Møller and L. Rosenfeld, Kgl. Danske Vid. Sels. Math.-Fys. Med. **17**, No. 8 (1940).

[10] E. C. G. Stueckelberg, Helv. Phys. Acta **13**, 347 (1940).

[11] R. Serber and S. M. Dancoff, Phys. Rev. **62**, 85 (1942).

multiplied by the imaginary unit i in the quantum mechanical interpretation, we have in both cases

$$[\varphi(x), \varphi(x')]=[\pi(x), \pi(x')]=0,$$
$$[\pi(x), \varphi(x')]=\delta(x-x'); \tag{3}$$

$$[\sigma_i, \sigma_j]=-2\sigma_k, \tag{4}$$

i, j, k cyclic permutation of 1, 2, 3,

and for the time derivative $\dot{F}$ of every observable F holds the equation of motion

$$\dot{F}=[H, F]. \tag{5}$$

In quantum theory σ_1, σ_2, σ_3 are the well-known spin matrices, while in the classical theory we can assume $\boldsymbol{\sigma}$ to be a unit vector.

For the purpose of our generalization it is convenient to pass from the ordinary space to the momentum space with the help of the Fourier transformation

$$\varphi(x)=(2\pi)^{-\frac{1}{2}}\int q(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \tag{6a}$$

$$\pi(x)=(2\pi)^{-\frac{1}{2}}\int p(k)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k},$$

$$q(k)=(2\pi)^{-\frac{1}{2}}\int \varphi(x)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}, \tag{6b}$$

$$p(k)=(2\pi)^{-\frac{1}{2}}\int \pi(x)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.$$

The rule (3) for the brackets is then equivalent to

$$[q(k), q(k')]=[p(k), p(k')]=0,$$
$$[p(k), q(k')]=\delta(k-k'). \tag{7}$$

We further introduce the Fourier-transformed function $v(k)$ of $U(x)$ according to

$$U(x)=(2\pi)^{-3}\int v(k)e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{k}, \tag{8}$$

$$v(k)=\int U(x)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}.$$

The different choice of the normalization factors in (8) is convenient because the condition (2) is then equivalent to the simple statement

$$v(0)=1. \tag{9}$$

The Hamiltonian written in the momentum space is given by

$$H=\tfrac{1}{2}\int\{p(k)p(-k)+k_0^2q(k)q(-k)\}d\mathbf{k}$$
$$+\frac{if}{\pi\sqrt{2}}\int v(-k)\boldsymbol{\sigma}\cdot\mathbf{k}q(k)d\mathbf{k}, \tag{10}$$

where

$$k_0=+(k^2+\mu^2)^{\frac{1}{2}}. \tag{11}$$

In order to fulfill the conditions that $U(x)$, $\pi(x)$, $\varphi(x)$ have to be real, the quantities $p(k)$, $q(k)$, $v(k)$ have to fulfill the reality conditions

$$q(-k)=q^*(k), \quad p(-k)=p^*(k),$$
$$v(-k)=v^*(k), \tag{12}$$

where a star denotes the conjugate complex. It is, however, remarkable that the reality of the Hamiltonian (and also of the total momentum) holds already, if the weaker conditions,

$$v(k)q(-k)=v^*(-k)q^*(k),$$
$$v(k)p(k)=v^*(-k)p^*(-k),$$
$$p(k)p(-k)=p^*(k)p^*(-k), \tag{13}$$
$$q(k)q(-k)=q^*(k)q^*(-k),$$
$$v(k)v(-k)=v^*(k)v^*(-k),$$

are fulfilled. The latter conditions are simplified by using the new variables

$$\tilde{q}(k)=q(k)v(-k), \quad \tilde{p}(k)=p(k)v(k) \tag{14}$$

which satisfy reality conditions analogous to (13),

$$\tilde{q}(-k)=\tilde{q}^*(k), \quad \tilde{p}(-k)=\tilde{p}^*(k)$$

and by using the real quantity

$$G(k)=G^*(k)=v(k)v(-k). \tag{14a}$$

The Hamiltonian in the new variables is

$$H=\tfrac{1}{2}\int[G(k)]^{-1}\{\tilde{p}(k)\tilde{p}(-k)+k_0^2\tilde{q}(k)\tilde{q}(-k)\}d\mathbf{k}$$
$$+\frac{if}{\pi\sqrt{2}}\int\boldsymbol{\sigma}\cdot\mathbf{k}\tilde{q}(k)d\mathbf{k}. \tag{15}$$

The value of the brackets is

$$[\tilde{p}(k), \tilde{q}(k')] = G(k)\delta(k-k'). \tag{16}$$

The new form of the Hamiltonian makes it evident that models with different source functions $v(k)$ belonging to the same function $G(k)$ are equivalent, and that only the latter function has a physical meaning. According to (9) one has always

$$G(0) = 1, \tag{17}$$

while the particular case $G(k) = \text{const.} = 1$ corresponds to a point source.

Whereas according to the stronger reality conditions (13) of the usual model of an extended source, $G(k)$ is necessarily positive, the renouncement of the reality of the field variables in the ordinary space has led us to the weaker condition that $G(k)$ has to be real only.

It has to be shown that the particular choice

$$G(k) = \cos(\lambda_0 k_0 - \boldsymbol{\lambda}\cdot\mathbf{k}) \tag{18}$$

has the remarkable property that an originally Lorentz invariant theory remains so if λ_0, $\boldsymbol{\lambda}$ is simultaneously transformed as a four vector. Finally, one has to carry through the so-called "λ-limiting process," namely, $(\lambda_0, \boldsymbol{\lambda}) \to 0$, which eliminates the particular choice of the λ vector from the final results and which makes all divergences of the classical model disappear provided that the four vector λ_0, $\boldsymbol{\lambda}$ is always timelike, that is, if

$$\lambda_0^2 > \boldsymbol{\lambda}^2. \tag{19}$$

As long as one has only to consider a single coordinate system, it is permissible to put $\boldsymbol{\lambda} = 0$; hence

$$G(k) = \cos\lambda_0 k_0 = \cos\{\lambda_0(k^2+\mu^2)^{\frac{1}{2}}\}. \tag{18a}$$

The λ-limiting process avoids the (classical) divergences of the point-source model without introducing in the final results a finite extension of the source and without destroying the relativistic invariance of a theory.

In earlier papers we defined for the model of an extended source the reciprocal of its radius a by

$$a^{-1} = \int\int \frac{U(x)\,U(x')}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}d\mathbf{x}'$$

which in the momentum space can be written

$$a^{-1} = \frac{1}{2\pi^2}\int G(k)\frac{d\mathbf{k}}{k^2}. \tag{20}$$

Maintaining this definition of a^{-1} also in the case of a non-positive $G(k)$ without any longer connecting a with the dimension of the source, we find for the particular choice (20) for $G(k)$[12]

$$a^{-1} = \frac{2}{\pi}\int_0^\infty \cos\{\lambda_0(k^2+\mu^2)^{\frac{1}{2}}\}dk = \mu J_1(\lambda_0\mu). \tag{21}$$

It is remarkable that the limiting process applied to the quantity a^{-1} gives zero:

$$\lim_{\lambda_0\to 0} a^{-1} = 0; \tag{21a}$$

while for the original model of a finite source, the transition to a point source, namely, $G(k)\to 1$ means always $a^{-1}\to\infty$. Some physical consequences of the result (21a) are given in the next section.

3. APPLICATION TO THE CLASSICAL THEORY OF PROTON ISOBARS (FREE GYRATION)

As is well known there exists in the classical interpretation of the theory, under certain conditions for the value of the coupling constant and for the source function, the possibility of a continuum of energy values for a heavy particle surrounded by its own meson field due to the presence of a free gyration of the field and the spin or isotopic spin of the heavy particle. This type of periodic motion was interpreted by Oppenheimer and Schwinger[13] as corresponding to discrete energy values in the quantum-theoretical treatment in which the total charge and the total angular momentum of the system have discrete values which are in suitable units integers and half-odd integers, respectively. They proved also that in the case of a strong coupling the treatment according to the correspondence principle agrees with the more rigorous quantum-mechanical treatment in the case of the charged scalar and the neutral pseudoscalar theory.[14] It

[12] Compare for the evaluation of the integral, J. M. Jauch, Phys. Rev. **63**, 334 (1943), Appendix.

[13] J. R. Oppenheimer and J. Schwinger, Phys. Rev. **60**, 150 (1942).

[14] The quantum-mechanical treatment in the case of the charged and symmetrical pseudoscalar theory was given by S. M. Dancoff and W. Pauli, Phys. Rev. **62**, 85 (1942).

is the purpose of the following computation to investigate the conditions for the existence of such excited states (isobars) of the heavy particle if the λ-limiting process is applied. It will be sufficient for it to give the classical treatment of the neutral pseudoscalar theory for the results in this case are typical also for more general cases.

We start with the equations of motion which one obtains according to the application of the general rule (5) to the Hamiltonian (15) if one inserts for F the quantities $\tilde{\varphi}(k)$, $\tilde{q}(k)$ and the components σ_1, σ_2, σ_3 of the spin of the heavy particle using the bracket expressions (4) and (16). In this way one obtains

$$\frac{\partial\tilde{q}(k)}{\partial t}=\tilde{p}(-k),$$

$$\frac{\partial\tilde{p}(k)}{\partial t}=-k_0^2\tilde{q}(-k)-\frac{if}{\pi\sqrt{2}}G(k)\boldsymbol{\sigma}\cdot\mathbf{k},$$

hence

$$\frac{\partial^2\tilde{q}(k)}{\partial t^2}+k_0^2\tilde{q}(k)=\frac{if}{\pi\sqrt{2}}G(k)\boldsymbol{\sigma}\cdot\mathbf{k}, \tag{22}$$

and

$$\dot{\boldsymbol{\sigma}}=-\frac{\sqrt{2}if}{\pi}\int[\boldsymbol{\sigma}\times\mathbf{k}]\tilde{q}(k)d\mathbf{k}. \tag{23}$$

We suppose a free gyration of the spin around the x_3 axis, according to

$$\sigma_1=(1-C^2)^{\frac{1}{2}}\cos\omega t,\quad \sigma_2=(1-C^2)^{\frac{1}{2}}\sin\omega t,\quad \sigma_3=C, \tag{24}$$

C being a constant between -1 and $+1$. Inserting this in (22) we obtain

$$\tilde{q}(k)=\frac{if}{\pi\sqrt{2}}G(k)\left\{\frac{(1-C^2)^{\frac{1}{2}}}{k^2+\mu^2-\omega^2}(k_1\cos\omega t+k_2\sin\omega t)+\frac{C}{k^2+\mu^2}k_3\right\}. \tag{25}$$

In the case

$$\omega^2<\mu^2 \tag{26}$$

the denominator in (25) is always different from zero and the field $\varphi(x)$ in the x space decreases to zero for large distances from the heavy particle. In the other case $\omega^2>\mu^2$ the field in large distances has the form of a spherical wave and describes free mesons not bounded by the heavy particle. In the following we shall therefore assume that the condition (26) holds.

By inserting the expression (25) for $\tilde{q}(k)$ in the right side of (23), the integrals can be simplified under the assumption that $G(k)$ is spherically symmetric; that means that it depends only on the absolute value k of the vector $\mathbf{k}$. In this case we obtain the following relation connecting the frequency ω with the values of C

$$\omega=\frac{4}{3\pi}f^2C\int_0^\infty G(k)\left\{\frac{1}{k^2+\mu^2-\omega^2}-\frac{1}{k^2+\mu^2}\right\}k^4dk. \tag{27}$$

Before we discuss this relation we write down the analogous expressions for the energy and for the angular momentum. The latter is given by

$$\mathbf{L}=-\pi\int[\mathbf{x}\times\boldsymbol{\nabla}\varphi]d\mathbf{x}+\tfrac{1}{2}\boldsymbol{\sigma},$$

or in the momentum space by

$$\mathbf{L}=\int[G(k)]^{-1}\tilde{p}(k)\left[\frac{\partial\tilde{q}(k)}{\partial\mathbf{k}}\times\mathbf{k}\right]d\mathbf{k}+\tfrac{1}{2}\boldsymbol{\sigma}. \tag{28}$$

It is in an integral of motion for a spherically symmetrical $G(k)$ a specialization which has already been mentioned. Inserting in the expression (15) for the Hamiltonian and in (28) for the angular momentum, the value (24) for $\boldsymbol{\sigma}$, (25) for $\tilde{q}(k)$ and $\dfrac{\partial\tilde{q}(-k)}{\partial t}$ for $p(k)$ one obtains for the energy

$$E=E_0-\frac{f^2}{3\pi}\int_0^\infty G(k)(1-C^2)\left\{\frac{1}{k^2+\mu^2-\omega^2}-\frac{1}{k^2+\mu^2}-\frac{2\omega^2}{(k^2+\mu^2-\omega^2)^2}\right\}k^4dk, \tag{29}$$

where

$$E_0=-\frac{f^2}{3\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2}dk \tag{29a}$$

is the self-energy of the nucleon corresponding to the ordinary static solution ($C=1$, $\omega=0$). The corresponding expression of the angular momentum, which in our problem has the direction of the x_3 axis ($L_1=L_2=0$, $L_3=L$) is

$$L=\frac{2f^2}{3\pi}\omega(1-C^2)\int_0^\infty G(k)\frac{k^4}{(k^2+\mu^2-\omega^2)^2}dk+\tfrac{1}{2}C. \tag{30}$$

The integrals occurring in (27), (29), (30) may be evaluated in the following way. We split the integrand into one part without denominator, containing the integrals

$$N=\frac{2}{\pi}\int_0^\infty G(k)k^2dk, \quad a^{-1}=\frac{2}{\pi}\int_0^\infty G(k)dk, \quad (31)$$

and another part which converges for a point source $G(k)\to 1$. In the latter part we shall insert $G(k)=1$ which is allowed both for the λ-limiting process and for an extended source [$G(k)$ positive] with dimensions small in comparison with μ^{-1}, that is, $\mu a \ll 1$. We shall restrict ourselves to these two cases. It has to be emphasized, however, that in the former case for which our formulas will also hold, μa is in no way small. Using

$$\frac{2}{\pi}\int_0^\infty \frac{dk}{k^2+\mu^2-\omega^2}=(\mu^2-\omega^2)^{-\frac{1}{2}},$$

$$\frac{2}{\pi}\int_0^\infty \frac{dk}{(k^2+\mu^2-\omega^2)^2}=\tfrac{1}{2}(\mu^2-\omega^2)^{-\frac{1}{2}},$$

and its specializations for $\omega=0$, one finds with the help of the indicated method,

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2}dk=N-\frac{\mu^2}{a}+\mu^3,$$

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{k^2+\mu^2-\omega^2}dk=N-\frac{\mu^2-\omega^2}{a}+(\mu^2-\omega^2)^{\frac{1}{2}},$$

$$\frac{2}{\pi}\int_0^\infty G(k)\frac{k^4}{(k^2+\mu^2-\omega^2)^2}dk=\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}.$$

Equations (27), (29), (30) get in this way their final form

$$\omega=\frac{2}{3}f^2C\left\{\frac{\omega^2}{a}-[\mu^3-(\mu^2-\omega^2)^{\frac{1}{2}}]\right\}, \quad (32)$$

$$E=E_0+\frac{f^2}{6}(1-C^2)$$

$$\times\left\{\frac{\omega^2}{a}+\mu^3-(\mu^2-\omega^2)^{\frac{1}{2}}-3\omega^2(\mu^2-\omega^2)^{\frac{1}{2}}\right\}, \quad (33)$$

$$E_0=-\frac{f^2}{6}\left(N-\frac{\mu^2}{a}+\mu^3\right), \quad (33a)$$

$$L=\frac{1}{3}f^2\omega(1-C^2)\left\{\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}\right\}+\frac{1}{2}C. \quad (34)$$

For periodic motions the energy of which depends only on the amount of the resulting angular momentum, the frequency ω is given by the general mechanical relation

$$\omega=dE/dL. \quad (35)$$

Introducing the auxiliary function of the frequency

$$W=\omega L-(E-E_0), \quad (36)$$

this relation is equivalent to

$$dW/d\omega=L. \quad (37)$$

The latter form can be checked easily in our case. We obtain first

$$W=\frac{1}{2}\omega C+\frac{f^2}{6}(1-C^2)\left\{\frac{\omega^2}{a}-\mu^3+(\mu^2-\omega^2)^{\frac{1}{2}}\right\}, \quad (38)$$

and we note that according to (32) this can also be written

$$W=\frac{1}{2}\omega\left(C+\frac{1-C^2}{2C}\right)=\frac{\omega}{4C}. \quad (38a)$$

From (38) one derives

$$dW=dC\frac{1}{2}\left\{\omega-\frac{2}{3}f^2C\left[\frac{\omega^2}{a}-\mu^3+(\mu^2-\omega^2)^{\frac{1}{2}}\right]\right\}$$

$$+d\omega\left\{\frac{1}{2}C+\frac{1}{3}f^2\omega(1-C^2)\left[\frac{1}{a}-\frac{3}{2}(\mu^2-\omega^2)^{\frac{1}{2}}\right]\right\}.$$

The factor of dC vanishes according to (32), whereas the factor of $d\omega$ is equal to L according to (34); hence the relation (37) is proved.

The terms proportional to $1/a$ in the condition (32) for the frequency can easily be interpreted. By inserting (25) in (24) it follows that the terms proportional to $1/a$ in the equation of motion for $\boldsymbol{\sigma}$ are

$$\dot{\boldsymbol{\sigma}}=-\tfrac{3}{8}f^2a^{-1}[\boldsymbol{\sigma},\ddot{\boldsymbol{\sigma}}]+\cdots. \quad (39)$$

On account of the linear connection between $\tilde{q}(k)$ and $\boldsymbol{\sigma}$ [Eq. (22)] Eq. (39) holds generally,

and not only for periodic motions as would appear from its derivation. Bhabha[15] in his classical treatment of the neutral vector mesons called the expression on the right-hand side of (39) the mechanical spin inertia term. The result $a^{-1}=0$ of the λ-limiting process means therefore that this process makes the spin inertia disappear; hence this process is in accordance with Bhabha's conjecture that for the real particles of nature the inertia terms will always be zero.

We add the remark that for $G(k)=\cos\lambda_0 k_0$, not only a^{-1} does tend to zero together with λ_0 but also N defined in (31), and according to (33a) the self-energy of the nucleon then becomes $E_0=-\frac{1}{6}f^2\mu^3$ which is small in comparison with the proton mass for the actual value of the coupling constant $(f\mu)^2\sim\frac{1}{10}$.

We now turn to the discussion of whether Eq. (32) for ω has a solution for $-1\leqq C\leqq 1$ and $\omega<\mu$. We carry through the discussion for two cases. First we assume a small extended source corresponding to a positive $G(k)$ and to $\mu a\ll 1$. In this case it is sufficient to retain the terms proportional to a^{-1}, and we obtain for the frequency

$$\omega=\frac{3}{2f^2}\frac{a}{C}$$

which can be fulfilled in the intervals in question for

$$(f\mu)^2>\tfrac{3}{2}(\mu a). \tag{40}$$

With the same approximation we obtain from (33), (34), (38):

$$\begin{gathered} W=\tfrac{1}{2}\omega L=E-E_0,\quad L=1/2C,\\ \omega=\frac{3a}{f^2}L,\quad E-E_0=\frac{3a}{2f^2}L^2. \end{gathered} \tag{41}$$

The condition $\omega<\mu$ implies an upper bound for L. This is the old result of Oppenheimer and Schwinger which in the case of a strong coupling $(f\mu)^2\gg\mu a$ agrees with the quantum mechanical result if we put $L^2=l(l+1)$ with a half-odd integer l which is not too large.

The second case we want to discuss is the result of the λ-limiting process with $a^{-1}=0$. Putting $x=\omega/\mu$ Eq. (32) has here the form

$$-1=\tfrac{2}{3}(f\mu)^2C\frac{1-(1-x^2)^{\frac{1}{2}}}{x},$$

which has solutions for $0\leqq x\leqq 1$ and $-1\leqq C\leqq 1$ only if

$$\gamma\cdot\tfrac{2}{3}(f\mu)^2>1, \tag{42}$$

where γ is the maximum of the function $[1-(1-x^2)^{\frac{1}{2}}]x^{-1}$ in the interval $0\leqq x\leqq 1$ which is only slightly larger than 1. This is not fulfilled in nature, where $(f\mu)^2\sim\frac{1}{10}$. Therefore in the theory based on the λ-limiting process there does not exist a free gyration corresponding to a stable excited state of the nucleon with the actual value of the coupling constant, a result stated in the introduction.

The theory of the scattering of a free meson can be treated in a similar way with this classical model. We do not give, however, this calculation in this paper because it contains nothing new in comparison with the analogous computation of Bhabha.[16] The quantum theory of the meson scattering, especially the comparison of the results of the theory based on the λ-limiting process with the theory of Heitler and Peng,[17] needs further investigation.

4. THE HAMILTONIAN FOR A MIXED PSEUDOSCALAR AND VECTOR FIELD IN INTERACTION WITH SEVERAL NUCLEONS

We must now generalize our Hamiltonian by including the isotopic spin, assuming the presence of several heavy particles and considering a mixture of a pseudoscalar meson and a vector meson. In the following we describe the nucleons, not with the help of a wave field in ordinary space and second quantization, but with the configuration space. The coordinates of the nucleons are denoted by $\mathbf{z}_A$ where capital Roman indices enumerate the different nucleons and run from 1 to N if N nucleons are present. As in the last section we introduce in the interaction energy

[15] See reference 6. In our units Bhabha's constant K is connected with our a^{-1} by $K/I=\frac{3}{4}f^2a^{-1}$.

[16] H. J. Bhabha, reference 6, p. 333 ff. One has to identify his constant g_2^2/I with our f^2 and his constant β with our a^{-1}. The λ-limiting process makes $\beta=0$. The total scattering coefficient of the meson by the nucleon is identical with Bhabha's expression (82), p. 337, if one substitutes for $\sin^2\theta$ its average $\frac{2}{3}$ and includes a factor 2 which is due to the difference between the pseudoscalar theory here considered and the vector theory treated by Bhabha.

[17] W. Heitler and H. W. Peng, Proc. Camb. Phil. Soc. **38**, 296 (1942); W. Heitler, Proc. Camb. Phil. Soc. **37**, 291 (1941).

of the nucleons with the meson field first a general source function $U(x)$ which degenerates into a δ function for a point source and which will later be specialized according to the λ-limiting process which re-establishes the relativistic invariance of the theory.

The total Hamiltonian of the system consists of three parts H_S, H_V, H_M which describe the pseudoscalar mesons, the vector mesons, and the nucleons, respectively. For the sake of simplicity we neglect the interaction of the longitudinal vector mesons with the nucleons, and retain only the interaction of the transverse vector mesons and the pseudoscalar mesons. If we denote by α, β the Dirac matrices and put as usual

$$\gamma=-i\beta\alpha;\quad s_1=-i\alpha_2\alpha_3,\ \cdots;\qquad \gamma_5=-i\alpha_1\alpha_2\alpha_3; \tag{43}$$

and denote with τ_α ($\alpha=1, 2, 3$) the components of the isotopic spin, one has according to Kemmer[18] and Bhabha[19] for the "symmetric" theory

$$\begin{aligned}H_S=&\tfrac{1}{2}\int\sum_\alpha\{\pi_\alpha^2+(\nabla\varphi_\alpha)^2+\mu^2\varphi_\alpha^2\}d\mathbf{x}\\&+(4\pi)^{\frac{1}{2}}f_S\int\sum_{A,\alpha}U(x-z_A)\tau_\alpha^A\\&\qquad\times\{\mathbf{s}^A\cdot\nabla\varphi_\alpha-\gamma_5^A\pi_\alpha\}d\mathbf{x}\\&+2\pi f_S^2\int\sum_{A,B,\alpha}U(x-z_A)\\&\qquad\times U(x-z_B)\tau_\alpha^A\tau_\alpha^B\gamma_5^A\gamma_5^Bd\mathbf{x}+,\end{aligned}\tag{44}$$

$$\begin{aligned}H_V=&\tfrac{1}{2}\int\sum_\alpha\Big\{\pi_\alpha^2+\frac{1}{\mu^2}(\nabla\cdot\pi_\alpha)^2\\&\qquad+(\nabla\times\boldsymbol{\phi}_\alpha)^2+\mu^2\boldsymbol{\phi}_\alpha^2\Big\}d\mathbf{x}\\&+(4\pi)^{\frac{1}{2}}f_V\int\sum_{A,\alpha}U(x-z_A)\tau_\alpha^A\\&\qquad\times\{\mathbf{s}^A\cdot(\nabla\times\boldsymbol{\phi}_\alpha)-\boldsymbol{\gamma}^A\cdot\pi_\alpha\}d\mathbf{x}\\&+2\pi f_V^2\int\sum_{A,B,\alpha}U(x-z_A)U(x-z_B)\\&\qquad\times\tau_\alpha^A\tau_\alpha^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x},\end{aligned}\tag{45}$$

$$H_M=\sum_A\Big\{\boldsymbol{\alpha}^A\cdot\frac{1}{i}\frac{\partial}{\partial\mathbf{z}^A}+M\beta^A\Big\}.\tag{46}$$

In the limiting case of a point source, the theory is relativistically invariant, φ being a pseudoscalar and $\boldsymbol{\phi}$ together with $\varphi_0=\mu^{-2}\nabla\cdot\boldsymbol{\pi}$ a four vector. π is canonically conjugate to φ and $\boldsymbol{\pi}$ canonically conjugate to $\boldsymbol{\phi}$ according to the bracket expressions ($\alpha, \beta=1, 2, 3$; $i, k=1, 2, 3$)

$$\begin{aligned}[\pi_\alpha(x),\ \varphi_\beta(x')]&=\delta_{\alpha\beta}\delta(x-x'),\\[\pi_{\alpha i}(x),\ \varphi_{\beta k}(x')]&=\delta_{\alpha\beta}\delta_{ik}\delta(x-x').\end{aligned}\tag{47}$$

Moreover, the components of the isotopic spin have the same commutation rules as the components of the ordinary spin, namely,

$$[\tau_1,\ \tau_2]=-2\tau_3,\ \cdots,\tag{48}$$

and the brackets of all quantities belonging to different nucleons are zero. With the help of these relations one obtains the equation of motion of all quantities by application of the general rule (5).

As was shown by Kemmer, the terms of the Hamiltonian quadratic in the coupling constants are not unique, but there exist the two possibilities to choose for them

$$2\pi f_S^2\int\sum_{A,B,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha^A\tau_\alpha^B\gamma_5^A\gamma_5^Bd\mathbf{x}$$

or

$$2\pi f_S^2\int\sum_{A,B,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha^A\tau_\alpha^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x}$$

for the pseudoscalar meson, and

$$2\pi f_V^2\int\sum_{A,B,\alpha}U(x-z_A)U(x-z_B)\tau_\alpha^A\tau_\alpha^B(\boldsymbol{\gamma}^A\cdot\boldsymbol{\gamma}^B)d\mathbf{x}$$

or

$$\begin{aligned}2\pi f_V^2\int\sum_{A,B,\alpha}&U(x-z_A)U(x-z_B)\\&\times\tau_\alpha^A\tau_\alpha^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{x}\end{aligned}$$

for the vector meson. The differences between these two alternatives are relativistic invariants in the limit of point sources.[20] There is no *a priori* reason for one possibility or the other because both can be derived from a Lagrangian without explicit terms of the second order in the coupling constant with suitable independent variables.

[18] N. Kemmer, Proc. Roy. Soc. **A166**, 127 (1938); we are interested in his cases (*c*) and (*d*) and in the interactions proportional to his constants f_b and g_d.

[19] H. J. Bhabha, Proc. Roy. Soc. **A166**, 501 (1938).

[20] With the help of quantized wave functions $\psi(z)$ and $\psi^+=\psi^*\beta$, $\mu,\ \nu=1,\ \cdots,\ 4,\ z_4=ix_0$, these invariants can be written $\sum_\mu[\psi^+i\gamma_5\gamma_\mu\psi]^2$ and $\sum_{\mu<\nu}[\psi^+i\gamma_\mu\gamma_\nu\psi]^2$, respectively.

The particular choice in the expressions (44) and (45) of these terms has only the *a posteriori* reason that it gives simpler results for the potential energy between two slowly moving nucleons, which in this case does not contain terms of the type $\delta(z_A-z_B)$. This situation, which has been noticed by different authors, does not seem entirely satisfactory.

The form of the Hamiltonians H_S and H_V as integrals over the momentum space is given by

$$\begin{aligned}H_S=&\tfrac{1}{2}\int\sum_\alpha\{p_\alpha(k)p_\alpha(-k)+k_0^2q_\alpha(k)q_\alpha(-k)\}d\mathbf{k}\\&+\frac{f_S}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha{}^A\\&\times\{i(\mathbf{s}^A\cdot\mathbf{k})q_\alpha(k)-\gamma_5{}^Ap_\alpha(k)\}d\mathbf{k}\\&+\frac{f_S{}^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k)\\&\times\tau_\alpha{}^A\tau_\alpha{}^B\gamma_5{}^A\gamma_5{}^B,\end{aligned}\tag{49}$$

$$\begin{aligned}H_V=&\tfrac{1}{2}\int\sum_\alpha\Big\{\mathbf{p}_\alpha(k)\cdot\mathbf{p}_\alpha(-k)\\&+\frac{1}{\mu^2}(\mathbf{k}\cdot\mathbf{p}_\alpha(k))(\mathbf{k}\cdot\mathbf{p}_\alpha(-k))\\&+k_0^2\mathbf{q}_\alpha(k)\cdot\mathbf{q}_\alpha(-k)\\&-(\mathbf{k}\cdot\mathbf{q}_\alpha(k))(\mathbf{k}\cdot\mathbf{q}_\alpha(-k))\Big\}d\mathbf{k}\\&+\frac{f_V}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha{}^A\\&\times\{i\mathbf{s}^A\cdot[\mathbf{k}\times\mathbf{q}_\alpha(k)]-\boldsymbol{\gamma}^A\cdot\mathbf{p}_\alpha(k)\}d\mathbf{k}\\&+\frac{f_V{}^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k)\\&\times\tau_\alpha{}^A\tau_\alpha{}^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{k}.\end{aligned}\tag{50}$$

The functions $p_\alpha(k)$, $q_\alpha(k)$ and $\mathbf{p}_\alpha(k)$, $\mathbf{q}_\alpha(k)$ are here defined as in Eq. (6), and they satisfy the canonical relation

$$\begin{aligned}[p_\alpha(k),q_\beta(k')]&=\delta_{\alpha\beta}\delta(k-k'),\\[p_{\alpha i}(k),p_{\beta j}(k')]&=\delta_{\alpha\beta}\delta_{ij}\delta(k-k'),\end{aligned}\tag{51}$$

and reality conditions analogous to (13). The functions $v(k)$ and $G(k)$ are again given by (8) and (15). Of course it is possible to define new quantities analogous to Eq. (14) of the preceding section which make it evident that only $G(k)$ has a physical meaning. This will not, however, be necessary for the following.

The part of H_V which describes the free mesons can be simplified by the canonical substitution

$$\begin{aligned}\mathbf{p}'&=\mathbf{p}-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p})+\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}),\\\mathbf{q}'&=\mathbf{q}-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q})-\frac{1}{\mu k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}),\end{aligned}\tag{52}$$

with the inverse formulae

$$\begin{aligned}\mathbf{p}&=\mathbf{p}'-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}')-\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}'),\\\mathbf{q}&=\mathbf{q}'-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}')-\frac{1}{\mu k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}').\end{aligned}\tag{52a}$$

One obtains in this way

$$\begin{aligned}H_V=&\tfrac{1}{2}\int\sum_\alpha\{\mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k)\\&+k_0^2\mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)\}d\mathbf{k}\\&+\frac{f_V}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\\&\times\tau_\alpha{}^A\Big\{is^A\cdot[\mathbf{k}\cdot\mathbf{q}_\alpha'(k)]\\&-\boldsymbol{\gamma}^A\cdot\Big[\mathbf{p}'(k)-\frac{1}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{p}'(k))\\&-\frac{\mu}{k^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{q}'(k))\Big]\Big\}d\mathbf{k}\\&+\frac{f_V{}^2}{4\pi^2}\int\sum_{A,B,\alpha}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k)\\&\times\tau_\alpha{}^A\tau_\alpha{}^B\beta^A\beta^B(\mathbf{s}^A\cdot\mathbf{s}^B)dk+\cdots.\end{aligned}\tag{50'}$$

In this paper we shall deal particularly with the nuclear forces derived from the so-called Rosenfeld-Møller mixture, which consists of a pseudoscalar and a vector meson with equal coupling constants. For the sake of simplicity

we shall assume also that the rest masses of the two kinds of mesons are equal. This may not be so in reality, but the purpose of the calculations carried through in the next section is not to compute the exact form of the nuclear forces, but merely to show the conditions for which the higher approximations of the perturbation theory regarding the nuclear forces are relatively small. For this purpose it may be sufficient to choose a simpler model than the actual case. Further simplifications arise from the approximation where the mass of the nucleon is considered infinite in comparison with the meson mass, or in other words where the recoil energy of the nucleon in all intermediate states can be neglected. In this case the four-component wave function of the nucleon can be reduced to a two-component wave function in a representation of the Dirac matrices where β is diagonal, the two smaller components being negligible. All matrices which anticommute with β, as for instance γ_5 and $\boldsymbol{\gamma}$, can be neglected. The four-component spin matrix $\mathbf{s}$ can be replaced by the two-component spin matrix $\boldsymbol{\sigma}$, and β can be put equal to unity. The part H_M of the Hamiltonian is then unnecessary, and putting in our case

$$f_S = f_V = g/\mu, \tag{53}$$

where g is dimensionless, one has

$$\begin{aligned} H = H_S + H_V = \tfrac{1}{2}\int \sum_\alpha \{ & p_\alpha(k)p_\alpha(-k) \\ & + \mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k) + k_0^2[q_\alpha(k)q_\alpha(-k) \\ & + \mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)]\}d\mathbf{k} \\ & + \frac{g}{\pi\sqrt{2}\mu}\int \sum_{A,\alpha} v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A)\tau_\alpha{}^A i\mathbf{s}^A \\ & \cdot\{\mathbf{k}q_\alpha(k) + [\mathbf{k}\times\mathbf{q}_\alpha'(k)]\}d\mathbf{k} \\ & + \frac{g^2}{4\pi^2\mu^2}\int \sum_{A,B,\alpha} \exp[i\mathbf{k}\cdot(\mathbf{z}_A - \mathbf{z}_B)] \\ & \times G(k)\tau_\alpha{}^A\tau_\alpha{}^B(\mathbf{s}^A\cdot\mathbf{s}^B)d\mathbf{k} + \cdots. \end{aligned} \tag{54}$$

As was shown by Stueckelberg,[21] the simple properties of this particular mixture appear more clearly if one applies the canonical substitution

$$\begin{aligned} \mathbf{q}'' &= (i/\mu)\{[\mathbf{k}\times\mathbf{q}'] + \mathbf{k}q\}, \\ \mathbf{p}'' &= -i(\mu/k^2)\{[\mathbf{k}\times\mathbf{p}'] + \mathbf{k}p\}, \\ q'' &= -(i/\mu)(\mathbf{k}\cdot\mathbf{q}'), \\ p'' &= i(\mu/k^2)(\mathbf{k}\cdot\mathbf{p}'), \end{aligned} \tag{55}$$

with the inversion

$$\begin{aligned} \mathbf{q}' &= -i(\mu/k^2)\{[\mathbf{k}\times\mathbf{q}''] + \mathbf{k}q''\}, \\ \mathbf{p}' &= (i/\mu)\{[\mathbf{k}\times\mathbf{p}''] + \mathbf{k}p''\}, \\ q &= i(\mu/k^2)(\mathbf{k}\cdot\mathbf{q}''), \\ p &= -(i/\mu)(\mathbf{k}\cdot\mathbf{p}''). \end{aligned} \tag{55a}$$

Moreover, the new functions fulfill the same reality conditions as the old ones. This substitution is convenient for the reason that the interaction energy depends only on $\mathbf{q}''$ but not on q''. Because of the relation

$$\begin{aligned} & p_\alpha(k)p_\alpha(-k) + \mathbf{p}_\alpha'(k)\cdot\mathbf{p}_\alpha'(-k) \\ & \quad + k_0^2[q_\alpha(k)q_\alpha(-k) + \mathbf{q}_\alpha'(k)\cdot\mathbf{q}_\alpha'(-k)] \\ & = \frac{k^2}{\mu^2}[\mathbf{p}_\alpha''(k)\cdot\mathbf{p}_\alpha''(-k) + p_\alpha''(k)p_\alpha''(-k)] \\ & \quad + \frac{k_0^2\mu^2}{k^2}[\mathbf{q}_\alpha''(k)\cdot\mathbf{q}_\alpha''(-k) + q_\alpha''(k)q_\alpha''(-k)] \end{aligned}$$

the field described by the new scalars p_α'', q_α'' does not interact with the nucleon and can be split off. We do not write it down any longer, and we omit the double prime again in the following final result:

$$\begin{aligned} H = & \frac{1}{2}\int \sum_\alpha \left\{ \frac{k^2}{\mu^2}\mathbf{p}_\alpha(k)\cdot\mathbf{p}_\alpha(-k) \right. \\ & \left. + \frac{k_0^2\mu^2}{k^2}\mathbf{q}_\alpha(k)\cdot\mathbf{q}_\alpha(-k)\right\}d\mathbf{k} \\ & + \frac{g}{\pi\sqrt{2}}\int \sum_{A,\alpha} v(-k)\exp(i\mathbf{k}\cdot\mathbf{z}_A) \\ & \times\tau_\alpha{}^A\boldsymbol{\sigma}^A\cdot\mathbf{q}_\alpha(k)d\mathbf{k} \\ & + \frac{g^2}{4\pi^2\mu^2}\int \sum_{A,B,\alpha} \exp[i\mathbf{k}\cdot(\mathbf{z}_A - \mathbf{z}_B)] \\ & \times G(k)\tau_\alpha{}^A\tau_\alpha{}^B(\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B). \end{aligned} \tag{56}$$

[21] E. C. C. Stueckelberg, see reference 10; the transformation is made there in the ordinary x space.

5. THE NUCLEAR FORCES BETWEEN NUCLEONS AT REST IN HIGHER APPROXIMATIONS FOR A ROSENFELD-MØLLER MIXTURE OF MESONS

The method of Stueckelberg to find the interaction energy between nucleons consists of the use of successive canonical transformations which eliminate step by step the field variables. The neglect of the motion (recoil energies in the intermediate states) of the nucleons is permitted as long as the distance between the nucleons is larger than their Compton wave-length M^{-1}.

Writing

$$H=H_0+gH_1+g^2H_2 \tag{57}$$

to express the dependence of the Hamiltonian on the coupling constant, we perform first a canonical transformation

$$H'=e^{iW}He^{-iW}=H+[W,H]+\frac{1}{2!}[W,[W,H]]+\cdots, \tag{58}$$

where H' is the Hamiltonian in the new variables. While the middle term $e^{iW}He^{-iW}$ holds only in quantum mechanics, the last form holds both in classical and quantum mechanics. We first assume W proportional to g according to $W=gW_1$ and arrange the result again in a power series of g

$$H'=H_0+g\{[W,H_0]+H_1\}+g^2\left\{\frac{1}{2!}[W_1,[W_1,H_0]]+[W_1,H_1]+H_2\right\}+g^3\left\{\frac{1}{3!}[W_1,[W_1,[W_1,H_0]]]+\frac{1}{2!}[W_1,[W_1,H_1]]+[W_1,H_2]\right\}+\cdots. \tag{59}$$

We choose W_1 in such a way that H_1 is just canceled, namely,

$$[W_1,H_0]=-H_1 \tag{60}$$

and obtain

$$H'=H_0+g^2\{\tfrac{1}{2}[W_1,H_1]+H_2\}+g^3\{\tfrac{1}{3}[W_1,[W_1,H_1]]+[W_1,H_2]\}+\cdots. \tag{61}$$

For the Hamiltonian (56) of the Rosenfeld-Møller mixture the condition (60) is fulfilled for

$$W_1=-\frac{1}{\pi\sqrt{2}}\int\sum_{A,\alpha}v(k)\exp(-i\mathbf{k}\cdot\mathbf{z}_A)\times\frac{k^2}{k_0^2\mu^2}\tau_\alpha{}^A\boldsymbol{\sigma}^A\cdot\mathbf{p}_\alpha(k)d\mathbf{k}.$$

The term $\frac{1}{2}[W_1,H_1]$ which occurs in H_2' consists of two parts of which one is independent of the field variables while the other is bilinear in this variable. One finds

$$\frac{g^2}{2}[W_1,H_1]=-\frac{g^2}{4\pi^2}\sum_{A,B,\alpha,\beta,i,j}\Bigg\{\tau_\alpha{}^A\tau_\alpha{}^B\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B\times\int\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]G(k)\frac{k^2}{k_0^2\mu^2}d\mathbf{k}+\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}\int\int v(k')v(-k)\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{z}_A]\times\frac{k'^2}{k_0'^2\mu^2}p_{\alpha i}(k')q_{\beta j}(k)d\mathbf{k}d\mathbf{k}'\Bigg\}. \tag{62}$$

Here the coefficient $\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}$ is an abbreviation for

$$\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}=[\tau_\alpha{}^A\sigma_i{}^A,\ \tau_\beta{}^A\sigma j^A]=-2\sigma_{ij}^A\delta_{\alpha\beta}-2\delta_{ij}\tau_{\alpha\beta}^A \tag{63}$$

(with the notation $\tau_{12}=-\tau_{21}\equiv\tau_3,\ \cdots;\ \sigma_{12}=-\sigma_{21}\equiv\sigma_3,\ \cdots$). Because of the particular choice of H_2 a certain cancellation occurs between the last term in (62) and H_2 since

$$\frac{1}{\mu^2}-\frac{k^2}{k_0^2\mu^2}=\frac{1}{k_0^2}$$

with the consequence that in the limit of point sources, $G(k)\to 1$, no interaction of the type $\delta(z_A-z_B)$ appears in the result. We get finally for the part $H'_{2,0}$ which is independent of the field variables—generally in the symbol $H'_{m,n}$ the first index is equal to the power of g with which it is multiplied, and the second index denotes the degree in the field variables—the result

$$g^2H'_{2,0}=\frac{g^2}{4\pi^2}\sum_{A,B,\alpha}\tau_\alpha{}^A\tau_\alpha{}^B\boldsymbol{\sigma}^A\cdot\boldsymbol{\sigma}^B\int\frac{G(k)}{k^2+\mu^2}\times\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}.$$

For $A \neq B$ it is always allowable to put $G(k)=1$. If we use

$$\frac{1}{2\pi^2}\int \frac{\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]}{k^2+\mu^2}d\mathbf{k}=\frac{\exp(-\mu r_{AB})}{r_{AB}}, \quad (64)$$

$r_{AB}=|\mathbf{z}_A-\mathbf{z}_B|$ being the distance between the two nucleons and take into account that every pair A, B occurs twice in the sum $\sum_{A,B}$ we find for the potential between two different nucleons A, B as far as it is proportional to g^2, the well-known result

$$V_{AB}=g^2\sum_\alpha \tau_\alpha{}^A\tau_\alpha{}^B\sigma^A\cdot\sigma^B[\exp(-\mu r_{AB})/r_{AB}]. \quad (65)$$

For $A=B$ we obtain the self-energy the value of which in the limit $G(k)\to 1$ is given by

$$V_{AA}=\frac{9}{2}g^2\left(\frac{1}{a}-\mu\right), \quad (66)$$

if, according to quantum theory, $\sum_\alpha \tau_\alpha{}^2=(\sigma^A)^2=3$ is inserted. For the case of the λ-limiting process we have to put $a^{-1}=0$; hence

$$V_{AA}=-\frac{9}{2}g^2\mu. \quad (66a)$$

In the discussion of terms of higher order in g we are mostly interested in the terms linear in the field variables, the lowest order of which is the third. These terms are according to (61), (62) given by

$$g^3H'_{3,1}=-\frac{g^3}{(2\pi^2)^{\frac{1}{2}}}\sum_{A,B,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\times\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}\times\int\exp(-i\mathbf{k}'\cdot\mathbf{z}_A)\frac{k'^2}{k_0{}'^2\mu^2}v(k')p_{\alpha i}(k')d\mathbf{k}',$$

where

$$\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\equiv\sum_{\beta,j}\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}\tau_\beta{}^B\sigma_j{}^B. \quad (67)$$

In the case $A=B$ one has to hermitize the right side by taking the arithmetical mean of the two orders of the factors $\begin{bmatrix}A\\ \alpha i,\beta j\end{bmatrix}$ and $\tau_\beta{}^A\sigma_j{}^A$. One finds in this way for $A=B$

$$\begin{bmatrix}AA\\ \alpha i\end{bmatrix}=0, \quad (67a)$$

for $A\neq B$

$$\begin{bmatrix}AB\\ \alpha i\end{bmatrix}=2\{(\sigma_k{}^A\sigma_l{}^B-\sigma_l{}^A\sigma_k{}^B)\tau_\alpha{}^B+\sigma_i{}^B(\tau_\beta{}^A\tau_\gamma{}^B-\tau_\gamma{}^A\tau_\beta{}^B)\} \quad (67b)$$

where (i, j, k) and (α, β, γ) are cyclic permutations of $(1, 2, 3)$.

These new linear terms in the field one has now to transform away by a new canonical transformation of the type (58) with the help of a $W=g^3W_3$ which satisfies the condition

$$[W_3, H_0]=-H'_{3,1}$$

and is given by

$$W_3=-\frac{1}{(2\pi^2)^{\frac{1}{2}}}\sum_{A,B,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\times\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}\times\int\exp(i\mathbf{k}'\cdot\mathbf{z}_A)\frac{k'^2}{k_0{}'^2\mu^2}v(-k')q_{\alpha i}(k')d\mathbf{k}'. \quad (68)$$

This transformation gives rise to a part $H''_{6,0}$ of the new Hamiltonian which is contained in $H_6''=\frac{1}{2}[W_3, H'_{3,1}]$. One finds[22]

$$g^6H''_{6,0}=\tfrac{1}{2}g^6\sum_{A,B,C,D,\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}CD\\ \alpha i\end{bmatrix}F_{CA}f_{AB}f_{CD} \quad (69)$$

with the definitions

$$f_{AB}=\frac{1}{2\pi^2}\int\frac{G(k)}{k_0{}^2}\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}, \quad (70)$$

$$F_{AB}=-\frac{1}{2\pi^2}\int G(k)\frac{k^2}{k_0{}^4\mu^2}\times\exp[i\mathbf{k}\cdot(\mathbf{z}_A-\mathbf{z}_B)]d\mathbf{k}. \quad (71)$$

For the limiting case of a point source the integral f_{AB} was evaluated above as

$$f_{AB}=\exp(-\mu r_{AB})/r_{AB} \text{ for } A\neq B, \qquad f_{AA}=a^{-1}-\mu. \quad (72)$$

[22] Compare E. C. C. Stueckelberg, reference 10, formulas (2.10) and (2.11).

In a similar way one finds

$$F_{AB}=\frac{1}{2\mu}\left(1-\frac{2}{\mu r_{AB}}\right)\exp(-\mu r_{AB}) \text{ for } A\neq B,$$
$$F_{AA}=\frac{1}{2\mu}\left(3-\frac{2}{\mu a}\right). \tag{73}$$

The result (69) means in general a three- and four-body interaction besides the two-body interaction. In the case of only two particles present the expression (69) gives an additional potential energy given by

$$V''_{AB}=\frac{g^6}{2\mu}\left[c_{AB}\frac{3\exp(-2\mu r_{AB})}{r_{AB}^2}+d_{AB}\left(1-\frac{2}{\mu r_{AB}}\right)\frac{\exp(-3\mu r_{AB})}{r_{AB}^2}\right], \tag{74}$$

where $A\neq B$ and

$$c_{AB}=\sum_{\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}AB\\ \alpha i\end{bmatrix},\quad d_{AB}=\sum_{\alpha,i}\begin{bmatrix}AB\\ \alpha i\end{bmatrix}\begin{bmatrix}BA\\ \alpha i\end{bmatrix}, \tag{75}$$

and where we put again $a^{-1}=0$ according to the λ-limiting process.

The evaluation of c_{AB} and d_{AB} with the help of the expression (67) gives, with convenient abbreviations, $s_{AB}\equiv(\sigma_A\cdot\sigma_B)$, $\tau_{AB}\equiv\sum_\alpha \tau_\alpha{}^A\tau_\alpha{}^B$:

$$c_{AB}=8\{3(3-s_{AB})+3(3-\tau_{AB})+4s_{AB}\tau_{AB}\}, \tag{76}$$

$$d_{AB}=-8\{(3-s_{AB})\tau_{AB}+s_{AB}(3-\tau_{AB})-4s_{AB}\tau_{AB}\}.[23] \tag{77}$$

[23] E. C. C. Stueckelberg, see reference 10, Eq. (3.12) gives in both c_{AB} and d_{AB} an additional term $-64(3-s_{AB})(3-\tau_{AB})$ which I have been unable to confirm and which seems to be erroneous. This additional term causes also an overestimation of the numerical value of V''_{AB} by Stueckelberg.

For the ground state of the deuteron one has $\tau_{AB}=-3$, $s_{AB}=1$, $c_{AB}=3.32$, $d_{AB}=-3.32$, and V''/V small for $3(2g)^4/(\mu r)^2\ll 1$ as was indicated in the introduction.

While in the classical theory the only terms of the final Hamiltonian which give rise to an interaction between nucleons are those which are independent of the meson field, it is different in quantum theory. In this theory it is not possible to put simultaneously $p_\alpha(k)$ and $q_\alpha(k)$ equal to zero, and terms in the final Hamiltonian which are either bilinear in the $p_\alpha(k)$ or bilinear in the $q_\alpha(k)$ give in general a finite contribution to the nucleon interaction due to the zero-point energy of the field oscillators. There exist such terms of the order g^4 which are generated by a g^2W_2 which eliminates $g^2H'_{2,2}$ given by the first term of (62), and which are computed by different authors. It is interesting that this interaction energy of the order g^4 disappears again according to the new theory of Dirac, where mesons of negative energy are introduced in the intermediate states. This theory brings forth a greater similarity with the classical theory than the older form of the quantum theory, and particularly all effects due to the zero-point fluctuations of the field oscillators are canceled in the new theory. It would be possible to check this new theory if experimental tests for or against the fourth-order terms of the interaction energy were available.

On the other hand this new hypothesis of negative-energy mesons was not necessary in order to make the theory convergent in the approximation where the heavy particles are treated non-relativistically. In this approximation the convergence could be achieved with the λ-limiting process alone.

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Note on the Production of Cosmic-Ray Mesons

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It is pointed out that the cross sections for the production of mesons in collisions between heavy particles given by Hamilton, Heitler, and Peng are so large that a heavy particle crossing an atomic nucleus must be expected to collide several times inside the same nucleus. This hypothesis accounts for cloud-chamber observations showing the production of several mesons in one point. Anticoincidence experiments of Jánossy and Rochester can also be interpreted easily by the theory of HHP when the occurrence of multiple collisions is assumed.

I. INTRODUCTION

IN a recent publication Hamilton, Heitler, and Peng[1] (quoted as HHP) investigate theoretically fast collisions between neutrons and protons. It is found that in fast collisions between heavy particles mesons are emitted. The cross section for the emission of mesons is found to be of the right order to account for most cosmic-ray phenomena. In the following we attempt to add to one point of the theory of HHP.

Cloud-chamber photographs of penetrating showers obtained by various observers[2–4] frequently show small groups of mesons diverging from a common center. These photographs are most easily interpreted on the assumption that mesons are produced simultaneously in small groups.

A typical photograph of this kind obtained recently by Jánossy, McCusker, and Rochester[4] is reproduced in Fig. 1. The photograph shows three tracks diverging from a point above the chamber. Each of the three tracks is passing through a lead plate, placed across the chamber, but none of the particles starts a cascade while passing through the plate. Thus the tracks cannot be assumed to be electrons. The simplest interpretation of the photograph is to assume that at least two of the three particles are mesons produced simultaneously in one point. The third particle is either a third meson or the primary particle which created the group.

The theory of HHP assumes that a fast heavy particle produces mesons one by one in succession. Multiple processes are mentioned as a possible source for the groups but are not dealt with by HHP. We wish to point out in the following that the groups of mesons observed can be accounted for in a simple way without considering multiple processes in the strict sense.

II. MULTIPLE COLLISIONS

The cross section for a fast neutron or proton to produce a meson with the energy greater than E in a collision with another heavy particle is according to HHP

$$\Phi(E)=1.5\times10^{-24}/E \text{ cm}^2. \quad (1)$$

All energies are measured in units of μc^2, where μ is the mass of the meson.

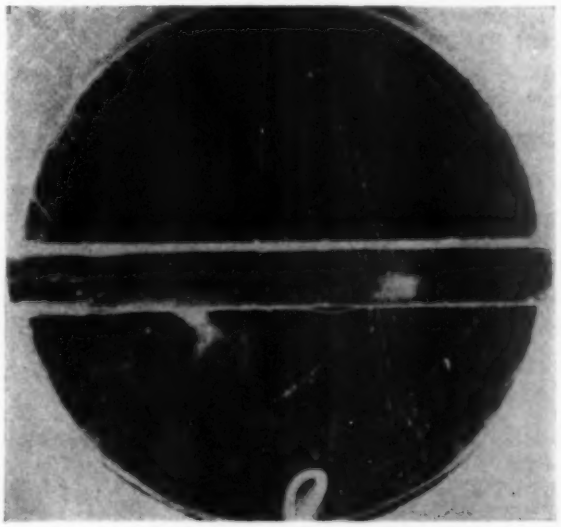

FIG. 1. Meson shower, showing three penetrating particles passing through a lead plate.

[1] Hamilton, Heitler, and Peng, Phys. Rev. **64**, 78 (1943).
[2] Braddick and Hensby, Nature **144**, 1012 (1939); G. Herzog and W. H. Bostick, Phys. Rev. **58**, 218 (1940).
[3] W. M. Powell, Phys. Rev. **58**, 474 (1940); **60**, 413 (1941).
[4] W. H. Bostick, Phys. Rev. **61**, 557 (1942); Jánossy, McCusker, and Rochester, Nature **148**, 660 (1941).

For moderately large values of the energy E the cross section given in (1) is so large that a fast heavy particle passing through a nucleus is likely to collide *several* times during the passage through the nucleus; several mesons may therefore be emitted in a collision with one nucleus. The collision of a fast heavy particle with a nucleus can be considered schematically as follows.

The radius of a nucleus of the weight A can be assumed to be $r_A=0.53(e^2/mc^2)A^{\frac{1}{3}}$ (compare Heisenberg[5]). The centers of the heavy particles forming the surface of the nucleus thus have an average distance $\bar{r}_A$ from the center of the nucleus given by (2),

$$\bar{r}_A=0.53(e^2/mc^2)(A^{\frac{1}{3}}-1). \tag{2}$$

Geometrically a collision between a constituent of the nucleus and the fast particle can take place when the fast particle passes at a distance less than $x=\bar{r}_A+r_E$ from the center of the nucleus, r_E being the radius of a disk with the area $\Phi(E)$ whence $\pi r_E^2=\Phi(E)$. The heavy particle passing through the nucleus has a chance to pass between the nuclear constituents without a collision. The probability for a passage without collision can be estimated roughly as

$$p=\exp\left[-\Phi(E)/(\pi\bar{r}_A^2/A)\right], \tag{3}$$

and therefore the effective cross section for the emission of at least one meson of the energy E in a collision with a nucleus of the weight A is estimated roughly as

$$\Phi(A, E)=(1-p)\pi(r_E+\bar{r}_A)^2. \tag{4}$$

Inserting numerical values one finds

$$\Phi(A, E)=1.5\times10^{-24}/E \times(1+aE^{\frac{1}{2}})^2(1-\exp\left[-b/E\right]). \tag{4a}$$

The constants a and b are functions of A only: Some values are given in Table I. Note that $\Phi(A=1, E)=\Phi(E)$ and that for large values of $E\Phi(A, E)\rightarrow A\Phi(E)$ as it must be expected.

TABLE I.

	Pb	Al	O	H
a	1.05	0.43	0.33	0
b	178	140	145	∞

[5] Heisenberg, Leipz. Ber. **89**, 369 (1937).

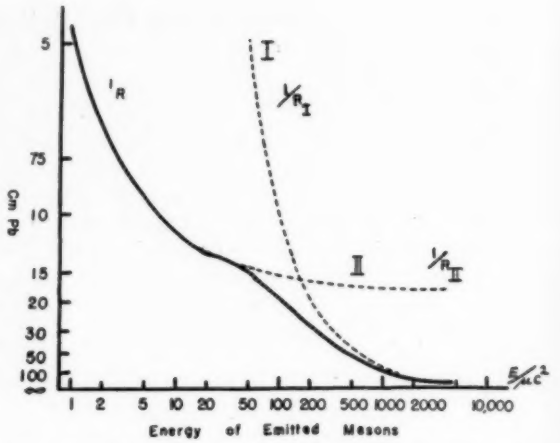

FIG. 2. Mean free path of fast heavy particles in lead. Curve I, $1/R_I$, packing neglected. Curve II, $1/R_{II}$, compact nucleus.

The reciprocal mean free path $1/R=N\Phi(A, E)$ as function of E for heavy particles in lead is plotted in Fig. 2 with the full line. N is the number of atoms per cc. The broken line marked I in Fig. 2 represents the reciprocal mean free path, $1/R_I=NA\Phi(E)$, one would expect if the packing of the nuclear particles could be neglected. The broken line marked II represents $1/R_{II}=N\pi(r_E+\bar{r}_A)^2$, the reciprocal mean free path one would expect if we neglect the possibility of a fast particle traversing a nucleus without encounter, i.e., regarding the nucleus as compact.

It is seen from Fig. 2 that for the emission of mesons with energies up to 5×10^9 ev ($E=50$), the lead nucleus can be regarded as compact, while for the emission of mesons of energies above 10^{11} ev ($E=1000$), the nuclear particles can be regarded as independent.

The average number n_A of mesons with energies exceeding E emitted by a sufficiently energetic fast particle while traversing a nucleus of weight A is expected to be

$$n_A=A\Phi(E)/\Phi(A, E) =A(1+aE^{\frac{1}{2}})^{-2}(1+\exp\left[-b/E\right])^{-1}. \tag{5}$$

The multiplicities thus obtained for lead are given in Table II. For lighter nuclei the multiplicities are somewhat smaller. The values obtained for n_A include pseudoscalar and transverse mesons as well as neutrettos.

The argument leading to Eqs. (4) and (5) assumes the nuclear particles to be independent.

This is of course not correct and in the accurate treatment interference would have to be taken into account. In the case of the emission of slow mesons the effect of interference may be serious. In the case of fast mesons, however, the de Broglie wave-lengths of the primary particle and of the emitted mesons are small compared with the average distances of the nuclear particles and the frequencies of both the primary particle and the emitted mesons are large compared with the nuclear frequencies. It is therefore likely that the results obtained on the assumption of independent nuclear particles are at least qualitatively correct for the emission of fast mesons.

We conclude therefore that the theory of HHP leads us to expect that mesons are produced in small groups.

III. COMPARISON WITH EXPERIMENTS

A. Cloud Chamber

The values of the multiplicities n_A given in Table II are, of course, only valid for primaries having sufficient energy to produce the required number of mesons. Primaries having less energy will be stopped completely inside the first nucleus with which they collide closely. The energy required to pass through a nucleus is of the order of 200. The energy required to pass through the atmosphere is according to HHP of the order of 10,000; the average energies of primaries which succeed in traversing the atmosphere must be expected to be of the same order. Thus it is expected that most of the primary particles reaching sea level have sufficient energy to traverse a lead nucleus.

The multiplicities n_A seem, however, to be too large to be compatible with the experimental evidence from cloud-chamber photographs. The predominance of small groups on photographs containing meson showers may be explained by assuming that most of the fast heavy particles near sea level are recoil particles produced by the primaries. The average energy of the recoil particles must be assumed to be small; and most of the recoil particles will have energies only sufficient to produce small groups.

In Fig. 3 a photograph of a big shower obtained by McCusker is reproduced. As the chamber was controlled with a heavy bias towards penetrating showers, it is not unlikely that this photograph represents a penetrating shower produced by a very energetic primary proton or neutron. The tracks seen on the photograph are, however, so numerous that it is impossible to decide whether or not some of the particles are mesons.

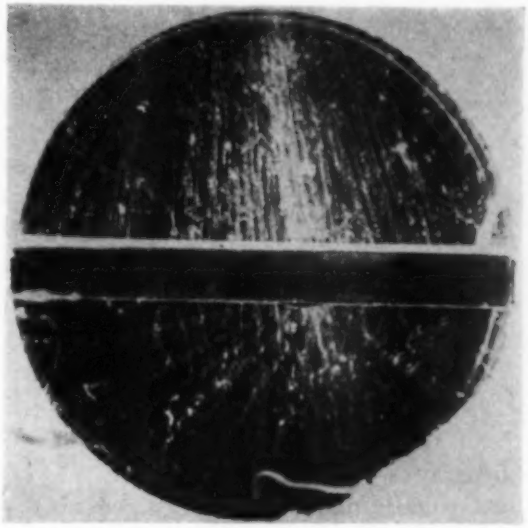

FIG. 3. Large shower obtained with an arrangement giving strong bias towards penetrating showers.

Shower photographs of Blackett and Occialini[6] and of Anderson[7] give evidence for the occurrence of slow heavy particles in showers. Jánossy, McCusker, and Rochester[4] have suggested that the occurrence of slow heavy particles is connected with penetrating showers. We note that the occurrence of slow heavy particles in penetrating showers is well accounted for by the theory of HHP in terms of slow recoil particles.

B. Counter Experiments

1. *Transition Effect of Penetrating Showers*[8]

Using a counter arrangement as shown in Fig. 4 we have recorded coincidences involving

TABLE II. Number of mesons with energy greater than E emitted by a fast heavy particle crossing a lead nucleus.

E	5	10	100	1000
n_A	18	11	1.7	1

[6] Blackett and Occialini, Proc. Roy. Soc. **A139**, 699 (1933).
[7] C. D. Anderson and S. H. Neddermeyer, Phys. Rev. **50**, 263 (1937).
[8] L. Jánossy and Ingleby, Nature **145**, 511 (1940); L. Jánossy, Proc. Roy. Soc. **A179**, 361 (1941).

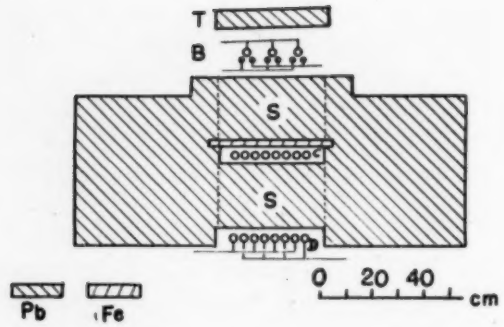

FIG. 4. Arrangement for observing penetrating showers.

the discharges of more than one counter out of each of the counter trays B, C, and D. The counter trays were separated by lead absorbers 15 cm thick, and the whole arrangement was surrounded with an absorber 50 cm thick. It was shown that most of the coincidences were due to penetrating showers. The rate of penetrating showers was found to increase when a lead absorber T was placed close above the top tray B. The increase was considerable for T a few cm thick, while no increase was found for thicknesses larger than five cm of lead. More detailed measurements of the transition effect will be reported shortly by Rochester and Jánossy.

2. Production of Penetrating Showers by Non-Ionizing Radiation (Jánossy and Rochester[9])

The arrangement is shown in Fig. 5. The lower part of the arrangement consisted of a coincidence arrangement sensitive to penetrating showers. The absorber T was surrounded with a set of anticoincidence counters A. Anticoincidences A–BCD, that is, coincidences BCD not accompanied by the discharge of any of the counters A, were recorded. These anticoincidences were largely due to penetrating showers started by non-ionizing particles falling on T.

To obtain information about the mean free path of the non-ionizing radiation giving rise to penetrating showers an absorber Σ was placed over the anticoincidence counters. It was found that five cm of lead placed above the counters A did not noticeably affect the anticoincidence rate. However, a lead absorber 35 cm thick placed over the anticoincidence counters cut down the rate of anticoincidences considerably. It was thus concluded that the mean free path of the non-ionizing radiation well exceeded 5 cm of lead. Thus the non-ionizing radiation could not be assumed to consist of photons, and it was suggested that it might consist of neutrons.

To avoid confusion we note that the term mean free path is used in the sense as the average distance a particle has to travel before encountering a collision of some specified kind. This mean free path is in general different from the range of a particle. If a particle is capable of suffering several collisions before being brought to rest, the range exceeds the mean free path considerably.

C. Interpretation of the Counter Experiments

The counter experiments described above were interpreted by HHP in terms of incident protons and neutrons. While we agree with this interpretation, we wish to point out that considerable difficulties are removed by introducing our hypothesis that a heavy particle collides several times in one nucleus.

1. If the collisions of a fast primary were distributed at random along its path and not crowded together into nuclei, then the mean free path for the production of mesons of energies greater than E could be obtained from Eq. (1) as

$$R_1 = 1/NA\Phi(E) = E/10 \text{ cm Pb}. \qquad (6)$$

Thus a fast particle traversing a few mm of lead would be expected to give rise to secondaries

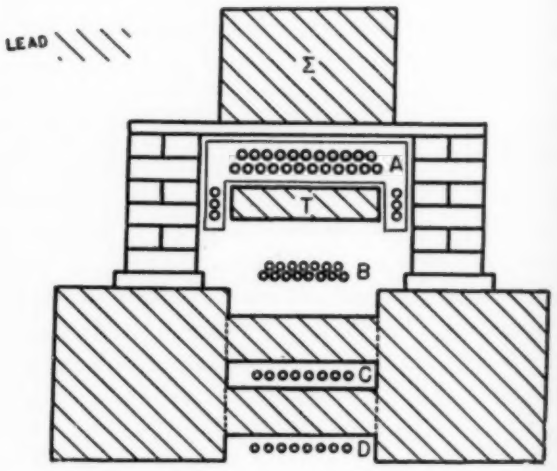

FIG. 5. Anticoincidence arrangement (Jánossy and Rochester).

[9] L. Jánossy and G. D. Rochester, Nature **150**, 633 (1942); Proc. Roy. Soc. **A** (in press); J. G. Wilson, Proc. Roy. Soc. **A174**, 73 (1940).

which would emerge together with the primary. This, however, is clearly incompatible with the interpretation of the anticoincidence experiment in terms of neutrons described in Section III, B2. The anticoincidence experiment shows that the non-ionizing particles have a good chance of penetrating five cm of lead without an encounter leading to ionizing secondaries emerging out of the lead.

Considering, however, the crowding of collisions, the mean free path of a primary has to be obtained by using Eq. (4). One finds

$$R=(1/N)\Phi(A, E). \qquad (7)$$

Inserting numerical values one sees that R is much greater than R_I. R, however, decreases rapidly with decreasing energy E, and the value R_0 of R corresponding to the mean free path as observed in the anticoincidence experiment is obtained as follows.

Collisions giving rise to mesons of energies so low that they are stopped inside the absorber do not prevent a neutron from giving rise to an anticoincidence.[10] Such collisions can therefore be neglected when considering anticoincidence experiments. Roughly one can neglect all collisions corresponding to values of E smaller than E_0 when the value of E_0 is chosen in such a way that the mean free path $R=R_0$ obtained from (7) for $E=E_0$ is equal to the range of mesons of the energy E_0. For lead one finds

$$E_0\sim 2, \quad R_0\sim 7 \text{ cm Pb} \qquad (8)$$

in good agreement with the experiment.

2. The transition effect of penetrating showers described in Section III, B1 can be interpreted in terms of protons and neutrons giving rise to penetrating showers in the absorber T. Saturation is expected for thicknesses somewhat exceeding R_0. This is compatible with the experimental results.

[10] A neutron, however, can be transformed into a proton while suffering a low energy collision, and thus it can be prevented from giving rise to an anticoincidence even if the mesons created in the collision remain inside the absorber. The loss of neutrons due to this process is, however, compensated by the gains of protons which are converted into neutrons in low energy collisions.

Comparing the results of the two experiments described in Section III, B, one finds that five cm of lead put into the path of the non-ionizing radiation has little effect on the anticoincidence rate, and therefore the mean free path of the neutral particles appears to be well above five cm of lead. Regarding the ordinary transition effect (Section III, B1), we note that most of the transition effect is over after five cm of lead. Thus it appears as if the ionizing radiation producing penetrating showers had a shorter mean free path than the non-ionizing radiation. If the two radiations consist of neutrons and protons as we think they do, this may indicate that the mean free path of protons is somewhat less than that of neutrons.

On the other hand the difference between the two experiments can probably be accounted for without assuming different mean free paths for protons and neutrons in the following way.

Penetrating showers are known to be accompanied by extensive air showers. A neutron falling on the anticoincidence arrangement (Fig. 5) may fail to produce an anticoincidence when accompanied by a sufficiently dense air shower as the air shower is likely to discharge some of the anticoincidence counters.

An absorber Σ five cm thick placed above the anticoincidence counters has a twofold effect. First, some of the neutrons which would be recorded without the absorber Σ give rise to mesons while passing through it and thus are not recorded. Secondly, some neutrons originally accompanied by air showers may be stripped of the shower while passing through Σ. Thus neutrons which are stripped of the accompanying cascade and happen to pass through Σ without encounter are recorded additionally because of the presence of Σ. The two effects may compensate each other to some extent, and it is quite likely that this accounts for the small absorption observed in the first five cm of lead.

I am indebted to Professor W. Heitler for valuable comment and for communicating his results, and also to Dr. G. D. Rochester for valuable discussion.

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Adiabatic Compressibilities of Some Aqueous Ionic Solutions and Their Variation with Indicated Liquid Structure of the Water*

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The adiabatic compressibilities of aqueous solutions of twenty-seven strong electrolytes, each at a single concentration, were determined at 25°C from sound velocity measurements at a supersonic frequency (1015.24 kc/sec.). The change in compressibility per mole fraction of solute shows a satisfactory correlation with x-ray and other indications of water structure change. Adiabatic compressibility values themselves show a remarkable correlation with corresponding values of the partial molal volume of the solvent water. Evidence strongly supports the conclusion that water has a liquid structure which becomes more highly coordinated and compacted with the introduction of ions.

INTRODUCTION

THIS report is upon the adiabatic compressibility of pure water and of aqueous solutions of twenty-seven strong electrolytes, and an interpretation of the results in terms of the liquid structure of the solvent water. Isothermal compressibility measurements by Amagat[1] and Bridgman[2] showed a minimum in the compressibility of water, and a dependence of compressibility values and the temperature of the minimum upon pressure. Numerous piezometric measurements have been made on water and its ionic solutions.[3] By modern supersonic methods, the adiabatic compressibilities of many organic liquids have been obtained.[4] In similar fashion, adiabatic compressibilities of a number of aqueous ionic solutions,[5] and of pure water over a great range of temperature,[6] have been precisely measured.

APPARATUS AND GENERAL METHOD

The method adopted for measurement involves the generation of a supersonic wave by a driven quartz crystal, and wave-length measurement by the optical method of secondary interferences.[7] The commercial x-cut quartz crystal plate, about 2.5 cm square and 2 mm thick, was driven indirectly by a radiofrequency power oscillator controlled by a quartz crystal. A single RK-20A tube was used in a conventional pentode circuit. The power supply was well filtered and regulated, and was more than adequate. A condenser-tuned resonant excitation control gave certain decided advantages: (a) control was smoothly variable over a wide range while leaving the power oscillator in operation under optimum conditions; (b) link coupling effectively insulated the control circuit and the crystal from the high supply voltages required by the power oscillator and permitted a convenient physical separation; (c) use of a current indicator permitted reproducible excitation values.

The driven quartz crystal was shellacked to a brass plate, which was mounted vertically within a brass box about 5.2 cm wide, 3.7 cm thick, and 6.5 cm high, to be itself submerged in the liquid being measured. The grounded supporting plate and a tinfoil sheet shellacked to the front face of the crystal supplied its electrodes. The emitting front surface of the crystal was made closely parallel to the plane front face of the crystal

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** Now of Sylvania Electric Products Inc., Flushing, Long Island, New York.
[1] E. H. Amagat, Comptes rendus **104**, 1159 (1887).
[2] P. W. Bridgman, Proc. Am. Acad. Arts Sci. **48**, 309 (1912).
[3] R. E. Gibson, J. Am. Chem. Soc. **59**, 1521 (1937); A. F. Scott, V. M. Obenhaus, and R. W. Wilson, J. Phys. Chem. **38**, 931 (1934); D. Tyrer, J. Chem. Soc. **105**, 2534 (1914).
[4] C. Bachem and E. Hiedemann, Zeits. f. Physik **94**, 68 (1935); E. B. Freyer, J. C. Hubbard, and D. H. Andrews, J. Am. Chem. Soc. **51**, 759 (1929); S. Parthasarathy, Proc. Ind. Acad. Sci. **2A**, 497 (1935).
[5] C. Bachem, Zeits. f. Physik **101**, 541 (1936); E. B. Freyer, J. Am. Chem. Soc. **53**, 1313 (1931); J. C. Hubbard and A. L. Loomis, Phys. Rev. **31**, 158 (1928); W. Schaaffs, Zeits. f. Physik **105**, 658 (1937).
[6] C. R. Randall, Bur. Stand. J. Research **8**, 79 (1932).
[7] C. Bachem, Zeits. f. Physik **101**, 541 (1936); C. Bachem and E. Hiedemann, Zeits. f. Physik **94**, 68 (1935); N. Seifen, Zeits. f. Physik **108**, 681 (1937–1938).

box, and lay about 6 mm directly behind the circular 2.4-cm aperture in the box face. The aperture was sealed with a thin mica diaphragm, which retained the xylol with which the crystal box was filled.

Actual measurement was made in a wave trough about 40 cm × 7 cm × 7 cm with the crystal box at one end and a plane brass reflector at the other. An interlocking brass framework around the open top of the trough provided pivot supports for the reflector and sliding ways for the support of the crystal box above the parallel plate glass sidewalls. Reflector orientation and crystal box position and orientation were controlled by propelling screws opposed by springs under tension. With the wave trough nearly filled with the liquid, with the driven crystal properly excited, and with correct orientation and separation of reflector and crystal box, standing waves were set up in the liquid, having a direction of propagation along the trough length.

Because standing density variations exist, the supersonic wave can be made visible by sending through it a transverse parallel beam of light, which is diffracted in very well-known fashion.[8] After emerging from such a diffracting medium, the light beam develops intensity variations[9] ("secondary interference"), which can be seen directly upon a screen placed in its path. The time average of the pattern as observed by the eye or recorded upon a photographic plate is that of the "sound grating" producing the diffraction: narrow vertical bright lines separated by the half wave-length ($\lambda/2$) of the supersonic wave in the liquid, or about $\frac{3}{4}$ mm in this research.

From a 110-volt, 15-amp., d.c. carbon arc, a condensing lens, and an adjustable vertical slit, a divergent light beam traversed the wave trough about 50 cm distant. Some 50 cm beyond the trough a small, shielded lens of focal length approximately 4.5 cm cast a magnified image of the light pattern on a screen about 20 cm away. Separation and orientation of crystal box and reflector were carefully adjusted to give a simple sharp pattern of vertical bright lines. Figure 1 gives a diagrammatic sketch of the optical system.

The wave trough was mounted on the sliding bearing-block of a Gaertner precision micrometer screw of half-millimeter pitch, which permitted a maximum excursion of about 6 cm. Displacements were read to 0.005 mm. When the trough was displaced in the direction of its length, the entire pattern moved across the screen with a spacial period of $\lambda/2$. It was found convenient to measure seventy half wave-lengths of the supersonic wave for each liquid. Such a method of measurement did not require parallel light, and permitted additional magnification of the light pattern.

The operating frequency was first measured by a General Radio Type 224 precision wave meter of the resonance type, which indicated approximately 1016 kc/sec. Then with a radio receiver tuned to KDKA, Westinghouse Station, Pittsburgh, Pennsylvania, an audible beat note was generated between the signal from the power oscillator, operating under load, and the carrier of KDKA. The frequency of the heterodyne represented the difference in frequency of the two radiofrequency waves and remained constant regardless of modulation of the commercial carrier.[10] Heterodyne frequency was measured as 4,760 c/sec. with a possible error of about 10 c/sec. Since KDKA maintains a carrier frequency within 2 c/sec. of its assigned 1020 kc/sec., the precise frequency used in this research was 1015.24 kc/sec., with a possible error of about 10 c/sec. at 25°C. Room temperature variations in the course of measurement were not important in introducing frequency change.

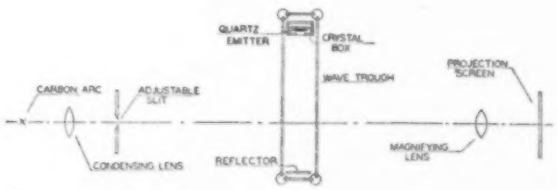

FIG. 1. Optical system.

From measured values of frequency and of distance $D = 70(\lambda/2)$ cm for the liquid used, the

[8] L. Bergmann, *Ultrasonics* (John Wiley & Sons, New York, 1938), English trans., p. 137; J. C. Hubbard, Am. J. Phys. **8**, 207 (1940); C. V. Raman and N. S. N. Nath, Proc. Ind. Acad. Sci. **2A**, 406 and 413 (1935).

[9] N. S. N. Nath, Proc. Ind. Acad. Sci. **4A**, 262 (1936).

[10] W. Bragg, J. Acous. Soc. Am. **11**, 5 (1939).

TABLE Ia. Summary of compressibility measurements at 25°C.

Substance	Mol. wt.	M. F.	V(m/s)	$10^{12}\beta$	$10^{12}\Delta\beta$	$-\Delta\beta/\Delta m \times 10^{12}$	$-\partial h/\partial m$
NH_4Cl	53.497	.01999	1543.31	41.389	3.329	1.665	1.18
$LiNO_3$	68.948	.005471	1509.17	43.512	1.206	2.204	1.6
$LiCl$	42.397	.01689	1551.14	40.767	3.951	2.339	1.38
$NH_4C_2H_3O_2$	77.064	.01040	1534.46	42.214	2.504	2.408	1.00
KCN	65.104	.009193	1525.76	42.340	2.378	2.587	.73
KCl	74.553	.009956	1526.05	41.993	2.725	2.737	2.25
$NaCl$	58.454	.01998	1564.77	39.230	5.488	2.747	1.48
$KClO_3$	122.553	.003075	1502.27	43.861	0.857	2.787	8.0
$CoSO_4$	155.004	.002395	1509.95	43.861	0.857	3.578	7.3
$NaOH$	40.005	.02090	1610.32	36.839	7.879	3.770	2.0
$Ca(NO_3)_2$	164.086	.003937	1509.81	42.875	1.843	4.681	2.2
$Mg(NO_3)_2$	148.336	.004041	1514.30	42.702	2.016	4.989	4.6
$MgCl_2$	95.234	.006012	1532.72	41.636	3.082	5.126	3.8
$Co(NO_3)_2$	182.956	.002006	1502.85	43.688	1.030	5.135	36.00
$NiCl_2$	129.604	.004117	1516.04	42.500	2.218	5.387	14.0
$CoCl_2$	129.854	.003289	1512.85	42.922	1.796	5.461	8.1
$(NH_4)_2SO_4$	132.144	.004004	1523.88	42.469	2.249	5.617	5.3
$CuSO_4$	159.634	.003946	1512.56	42.344	2.374	6.016	9.4
$AlCl_3$	133.341	.004656	1533.59	41.595	3.123	6.707	7.9
$MgSO_4$	120.384	.003107	1518.42	42.619	2.099	6.756	7.5
$K_2Cr_2O_7$	294.212	.001984	1504.30	43.368	1.350	6.804	16.0
Na_2CO_3	105.994	.01006	1589.14	37.531	7.187	7.144	No inf.
$Al(NO_3)_3$	212.994	.001950	1509.14	43.268	1.450	7.436	6.2
K_2SO_4	174.256	.003918	1526.43	41.798	2.920	7.453	9.5
$Co(C_2H_3O_2)_2$	176.988	.001754	1510.74	43.390	1.328	7.571	60.0
K_2CO_3	138.192	.003857	1530.11	41.757	2.961	7.677	4.6
$Ba(OH)_2$	171.386	.003125	1516.77	42.290	2.428	7.770	23.1
H_2O	18.016		1497.62	44.718			
Tap water			1497.62	44.703			

TABLE Ib. Summary of compressibility measurements at 25°C.

Substance	$\Delta D/\Delta t \times 10^3$	$\Delta v/\Delta t$ (cm/sec./°C)	Substance	$\Delta D/\Delta t \times 10^3$	$\Delta v/\Delta t$ (cm/sec./°C)
NH_4Cl	9.54	277	$CoCl_2$	8.77	254
$LiNO_3$	8.92	259	$(NH_4)_2SO_4$	8.91	258
$LiCl$	7.39	214	$CuSO_4$	8.00	232
$NH_4C_2H_3O_2$	8.00	232	$AlCl_3$	8.31	241
KCN	8.00	232	$MgSO_4$	9.08	263
KCl	10.77	312	$K_2Cr_2O_7$	9.23	268
$NaCl$	7.08	205	Na_2CO_3	8.46	245
$KClO_3$	9.69	281	$Al(NO_3)_3$	9.54	277
$CoSO_4$	11.23	326	K_2SO_4	9.08	263
$NaOH$	6.77	196	$Co(C_2H_3O_2)_2$	8.62	250
$Ca(NO_3)_2$	10.40	302	K_2CO_3	10.31	299
$Mg(NO_3)_2$	9.23	268	$Ba(OH)_2$	8.77	254
$MgCl_2$	7.39	214			
$Co(NO_3)_2$	8.92	259	H_2O	9.54	277
$NiCl_2$	9.39	272	Tap water	9.54	277

supersonic velocity is found from the relationship $v=\nu\lambda$ to be $v=0.029007D\times10^6$ cm/sec. Measurements by many investigators[11] show no certain dispersion of acoustic velocity in liquids for frequencies below about 10^9 c/sec.

Liquid density ρ was measured directly with a 25-cc pyknometer and sensitive analytical balances. Since the adiabatic compressibility $\beta=1/\rho v^2$, fractional error in β can be expressed:

$$\frac{\Delta\beta}{\beta}=-\left(\frac{\Delta\rho}{\rho}+\frac{2\Delta v}{v}\right)=-\left\{\frac{\Delta\rho}{\rho}+2\left(\frac{\Delta\nu}{\nu}+\frac{\Delta\lambda}{\lambda}\right)\right\}.$$

[11] A. K. Dutta, Physik. Zeits. **39**, 186 (1938); H. O. Kneser, Ann. d. Physik (Leipzig) **37**, 277 (1938); S. Parthasarathy, Proc. Ind. Acad. Sci. **4A**, 17 (1936); B. V. R. Rao and D. S. S. Ramiya, Phys. Rev. **60**, 615 (1941); B. Spakovskij, Comptes rendus Acad. Sci. U.R.S.S. (N.S.) **18**, 169 (1938).

The quantity $\Delta\lambda/\lambda$ is the largest contributor to $\Delta\beta/\beta$ and has a value less than 8×10^{-4}.

EXPERIMENTAL PROCEDURE. RESULTS

Solutions of the selected twenty-seven electrolytes were made with doubly-distilled water.[12] Each was at a single concentration ranging from a mole fraction of 0.02 for small molecular weights to 0.002 for the highest molecular weights used. Each solution, of approximately 1.6-liters volume, was carefully protected in a glass bottle with paraffin-coated cap. Contamination of the solutions during the course of measurement was largely prevented by coating with paraffin all surfaces excepting the glass sidewalls of the wave trough and the mica window of the crystal box, and by avoiding personal contact.

Temperature gradients in the liquid were effectively prevented by a motor-driven glass screw 2 cm in diameter and 8 cm long, operated at 4 or 5 r.p.s. in the wave trough outside the supersonic beam. However, no attempt was made to insulate the trough thermally or to preserve a constant temperature. Cooled at first to about 20°C, the liquid was allowed to change smoothly in temperature, during which time a number of quick determinations of wave-length and of density were made. Linear experimental curves of $D=70(\lambda/2)$ against t and of ρ against t were then constructed. From these curves, v and β were computed at 25°C. Measurements of temperature were made by an immersed thermometer.

Tables Ia and Ib present a summary of measurements at 25°C and a number of quantities obtained by further computation. In order to avoid error due to water of crystallization and moisture content of the solute crystals, P, the percent by weight of the solute, was determined for each solution by linear interpolation at a temperature for which both P and ρ are given in *International Critical Tables*, Vol. III. M.F. is the mole fraction of the solute. The symbol $\Delta\beta$ signifies $(\beta_{\text{water}}-\beta_{\text{solution}})$ at the temperature of measurement. The decrease in β per mole fraction

[12] The doubly-distilled water, prepared by W. W. Boardman of the State University of Iowa Chemistry Department, had an approximate conductivity of 5×10^{-6} mho/cm.

of the solute is given the symbol $-\Delta\beta/\Delta m$. The average increase in sound velocity per degree centigrade for each solution over the temperature range of 23°C to 26°C is given in Table Ib.

With any method of measuring sound velocity, excessive sound amplitude may cause errors. A high temperature gradient may exist in front of the emitting crystal. The emitted wave may not be plane, and in interferometric methods of measurement, successive positions of maximum reaction, chosen on the basis of average effect of all components, are not necessarily separated by the half wave-length.[13] It has been reported that the supersonic intensity required for velocity measurement with the supersonic interferometer for liquids is significantly less than the intensity required by optical methods;[14] but as can be seen from Table II, the sound intensity employed in this research has not been great enough to impair accuracy of measurement.

Within the limits of experimental errors, sound velocity is measured to be the same in tap water as in the most carefully prepared distilled water (Table Ia). Additional tests showed no velocity difference when ordinary distilled water was used, whether or not it had been freed of dissolved air by boiling. These results are in apparent disagreement with reports[15] which indicate a strong dependence of velocity upon gas content. It would seem that the discrepancy arises from two sources. First, experimental conditions under which the gas appears as fine bubbles throughout the liquid have not been carefully distinguished from conditions under which no bubbles appear; the liquid is clear and homogeneous; and the gas is truly dissolved. Second, since agitation of the liquid by the acoustic wave aids any tendency of the gas to come out of solution and to form bubbles, methods of measurement are subject to error if the sound intensity employed is so great that bubbles of the once-dissolved gas are formed, or actual cavitation of the liquids is produced.

In the present series of experiments the liquids remained homogeneous and clear. There was no apparent formation of gas bubbles in the body of the liquid due to excessive sound intensity. Under such conditions it is not surprising that no noticeable change in sound velocity was observed, since even water saturated with air at 25°C contains only two N_2 molecules and one O_2 molecule to approximately 2×10^5 molecules of water. Tap water from soft water sources may likewise have a low ionic concentration and an adiabatic compressibility which differs only slightly from the value for pure water.

DISCUSSION, INTERPRETATION, CORRELATION OF RESULTS

Precision compressibility measurements by Randall[16] have shown that with increasing temperature, the adiabatic compressibility for pure water decreases to a minimum about 64°C, then rises again. Not long after the early work of Amagat, Tamman[17] ennunciated a theory attributing the decreased compressibility of aqueous solutions to increased internal pressure. On the same basis, since increased external pressure was known to lower the temperature of minimum compressibility for water, he anticipated the same behavior for aqueous solutions due to increased internal pressure and supported the theory for the case of ethyl alcohol. Tamman's principle may be stated: "A given amount

TABLE II. Comparison of measured values of v and β for pure water.

Observer	Temp.	v (m/sec.)	$10^{12}\beta$	Date	Method	Freq. (kc/sec.)
Corey	25°C	1497.62	44.718	1941	Secondary interference	1015.24
Corey	24°C	1494.87	44.872	1941	Secondary interference	1015.24
Seifen	25°C	1497.15		1938	Secondary interference	
Schaaffs	17°C	1467		1937	Debye-Sears grating	
Bachem	25°C	1497.51	44.723	1936	Secondary interference	5023.3
Hiedemann, Seifen and Schreuer	25°C	1497.18		1936	Secondary interference	4900
Parthasarathy	24°C	1494	44.9	1935	Debye-Sears grating	7320
Bergmann	25°C	1465	47.98	1933	Debye-Sears grating	
Randall	25°C	1497.4		1932	Interferometer	767.64
Randall	20°C	1483.1		1932	Interferometer	767.64
Freyer, Hubbard, Andrews	24°C	1495		1928	Interferometer	
Hubbard and Loomis	25°C	1498.1		1928	Interferometer	437.6
Tyrer			45.31	1914	Piezometer	

[13] W. T. Richards, Rev. Mod. Phys. **11**, 44–48 (1939).
[14] W. Garten, Bull. Am. Phys. Soc. **16**, 7, Abstract 32 (1941).
[15] K. Dörsing, Ann. d. Physik **25**, 228 (1908); Ch. Sörensen, Ann. d. Physik (Leipzig) **26**, 121 (1936); Taylor and Sproule, Trans. Roy. Soc. Can. **3**, 23, 91 (1929).
[16] See reference 6.
[17] G. Tamman, Zeits. f. physik. Chemie **11**, 676 (1893).

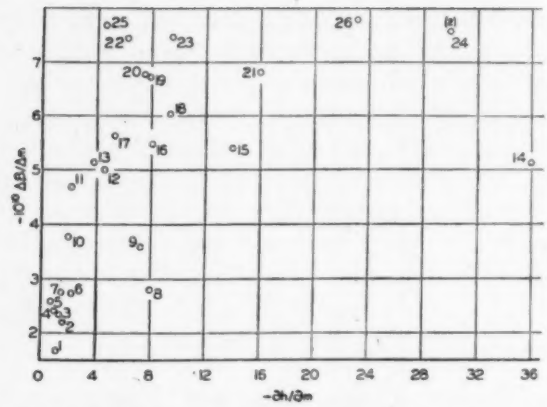

FIG. 2. Correlation between $-10^{10}\Delta\beta/\Delta m$ and $-\partial h/\partial m$ for the following numbered electrolytes: 1. NH_4Cl, 2. $LiNO_3$, 3. LiCl, 4. $NH_4C_2H_3O_2$, 5. KCN, 6. KCl, 7. NaCl, 8. $KClO_3$, 9. $CoSO_4$, 10. NaOH, 11. $Ca(NO_3)_2$, 12. $Mg(NO_3)_2$, 13. $MgCl_2$, 14. $Co(NO_3)_2$, 15. $NiCl_2$, 16. $CoCl_2$, 17. $(NH_4)_2SO_4$, 18. $CuSO_4$, 19. $AlCl_3$, 20. $MgSO_4$, 21. $K_2Cr_2O_7$, 22. $Al(NO_3)_3$, 23. K_2SO_4, 24. $Co(C_2H_3O_2)_2$, 25. K_2CO_3, 26. $Ba(OH)_2$.

of water in a solution behaves like the same amount of pure water subjected to some greater external pressure." A special series of measurements to 40°C on pure water and on a 5.6 percent solution of Na_2CO_3 gave results which appeared to be in accord with Tamman's principle, but showed definitely that even for maximum solute concentrations used, the compressibility minimum for the solution lay above even 50°C.

Bachem[18] and Freyer[19] have shown that the compressibility of an aqueous ionic solution decreases only slightly less than linearly with solute concentration over a great concentration range. Therefore, since concentrations used in the present research are 0.01 mole fraction or less, with a few exceptions of approximately 0.02 mole fraction, values of $-10^{10}\Delta\beta/\Delta m$ as they appear in Table Ia give a valid comparison of the behavior of different electrolytes. Since values of $-10^{10}\Delta\beta/\Delta m$ for the solutions are more or less characteristic of the solutes to which they pertain, a correlation is expected between these values and other mole fraction indications of change, such as a change in water structure as given by x-ray experiments. X-ray diffraction curves exhibiting maxima and minima, such as those for water and other liquids, may be produced by the existence of a few fairly definite interatomic distances.[20] From an experimental x-ray diffraction curve for a liquid, a unique radial distribution function may be directly determined, which yields information regarding number and arrangement of neighboring molecules, or the structure of the liquid. Changes in structure under different experimental conditions may thus be examined. But as might be expected, the diffraction curve itself shows an associated change. In the case of the x-ray diffraction curve for water, the second peak is particularly sensitive to changes in the radial distribution function, and by its change in height may serve as an indicator of the effects produced.[21] Introduction of ions, like an increase in temperature, has in general the effect of breaking down the structure and causing closer packing; and the second diffraction peak shows a correlated decrease in height as the coordination number increases. The quantity $-\partial h/\partial m$, proportional to the change in height of the peak per mole fraction of the solute, is an indication of the structural change characteristic of the solute.[22]

Figure 2 shows graphically the correlation between $-\partial h/\partial m$ and $-10^{10}\Delta\beta/\Delta m$ for twenty-six of the twenty-seven electrolytes considered in this research. Most of the observations lie within a rather small angle. The grouping of the univalent electrolytes, especially of $LiNO_3$, LiCl, $NH_4C_2H_3O_2$, KCN, KCl, and NaCl, indicates that the observed effects of an ion depend much more strongly upon its charge than upon its radius.

In general, a greater effect results from electrolytes of higher valence types. It is clear, however, that the effects of individual ions are not easily distinguishable. The behavior of an ion is much influenced by the character of the ion with which it is paired. For this reason, although one may arrange a sequence which is valid for a number of cases in a variety of experiments, one cannot set down unequivocally a series of cations or of anions in order of their effects.

A comparison of results shows satisfactory agreement with: conclusions of Passynski,[23] whose work was based upon a theory of Debye;[24]

[18] See reference 5.
[19] E. B. Freyer, J. Am. Chem. Soc. **53**, 1313 (1931).
[20] B. E. Warren, J. App. Phys. **8**, 645 (1937).
[21] J. Morgan and B. E. Warren, J. Chem. Phys. **6**, 666 (1938); G. W. Stewart, Phys. Rev. **37**, 9 (1931); G. W. Stewart, J. Chem. Phys. **7**, 869 (1939).
[22] G. W. Stewart, J. Chem. Phys. **7**, 869 (1939).
[23] A. Passynski, Acta Physicochimica (8) **4**, 385 (1938).
[24] P. Debye, *Festschrift H. Zangger* (Rasch and Co., 1935), Part II, 877.

the effects expected according to the theory of "permanent hydration," due to Bernal and Fowler;[25] the influence of ionic radius as given by Pauling;[26] and the evidence of change in water structure found by Frank and Robinson[27] in computed partial molal entropies. Because to a first approximation an ionic solution of low concentration may be looked upon as altered water, one expects a correlation between properties of the solution and properties of the water within the solution, i.e., the solvent water. From the same point of view it is of interest to compare at 25°C the values of solution compressibility, β, with corresponding values of $\bar{V}_1$, the partial molal volume of the solvent water. By use of the developments of Geffcken[28] and Wirth,[29] an expression has been derived for $\bar{V}_1$ which is applicable in all cases where Φ, the apparent molal ionic volume of the solute, varies linearly with some power of the solute concentration. For sixteen of the twenty-seven electrolytes used in this research, Φ does not vary linearly with c^1 (cf. Stewart[30]) as the simple Debye-Hückel theory predicts. The derivation of the expression follows immediately.

Explanation of symbols used:

V: volume of the solution in milliliters;
n: moles of designated substance in the solution;
c: moles of designated substance per liter of solution;
$\bar{V}=\partial V/\partial n$: partial molal volume of designated substance;
$\Phi(V)=\Phi$: apparent molal ionic volume of designated substance;
x: exponent of c which makes the plot of Φ against c^x a straight line;
k: slope of linear plot of Φ against c^x;
Subscript 1 designates the solvent;
Subscript 2 designates the solute;
Subscript or superscript 0 designates the value of a quantity when the solute concentration is zero.

If temperature and pressure remain constant, a small change in volume may be expressed generally as

$$dV=(\partial V/\partial n_1)dn_1+(\partial V/\partial n_2)dn_2. \quad (1)$$

If the mole ratio also remains constant:

$$dV/V=dn_1/n_1=dn_2/n_2=a. \quad (2)$$

Thus, upon substitution:

$$V=n_1(\partial V/\partial n_1)+n_2(\partial V/\partial n_2). \quad (3)$$

Since $\bar{V}_1=(\partial V/\partial n_1)$, etc., and $c_2=1000n_2/V$, etc.,

$$\bar{V}_1=\frac{V-n_2\bar{V}_2}{n_1}=\frac{1000-c_2\bar{V}_2}{c_1}. \quad (3a)$$

By definition,

$$\Phi(\bar{V}_2)=\Phi_2=\frac{V-V_0}{n_2}. \quad (4)$$

From (3), $V_0=n_1\bar{V}_1^0$; therefore

$$\Phi_2=\frac{1000-c_1\bar{V}_1^0}{c_2}, \text{ and } c_1=\frac{1000-\Phi_2c_2}{\bar{V}_1^0}.$$

Therefore from (3a)

$$\bar{V}_1=(\bar{V}_1^0 1000-c_2\bar{V}_2)/(1000-c_2\Phi_2). \quad (3b)$$

From (4), $V=V_0+n_2\Phi_2$; therefore,

$$(\partial V/\partial n_2)=\bar{V}_2=\Phi_2+n_2(\partial\Phi_2/\partial n_2)$$
$$=\Phi_2+n_2(\partial\Phi_2/\partial c_2)(\partial c_2/\partial n_2). \quad (5)$$

With the exponent x chosen empirically, the plot of Φ_2 against c^x may be made sensibly linear

TABLE III. Summary of measurements and computations on β and $\bar{V}_1$. The temperature in each case is 25°C.

Substance	Exp.	$\varphi(\bar{V}_2^0)$	k	c_1	$\bar{V}_1$	$10^{12}\beta$
NH_4Cl	0.5	36.30	1.225	1.0827	18.05642	41.389
$LiNO_3$	0.8	28.99	.605	0.3018	18.06839	43.512
LiCl	0.5	17.02	1.46	0.9342	18.05727	40.767
$NH_4C_2H_3O_2$	1.0	60.72	.74	0.5616	18.06503	42.214
KCN	1.0	30.5	.22	0.5055	18.06838	42.340
KCl	0.5	26.36	2.41	0.5480	18.06044	41.993
NaCl	0.5	16.28	2.22	1.1048	18.04563	39.230
$KClO_3$	1.0	45.78	−4.87	0.1694	18.07196	43.861
$CoSO_4$	1.0	−1.9	15.0	0.1329	18.06463	43.861
NaOH	1.0	−3.75	1.47	1.1840	18.03226	36.839
$Ca(NO_3)_2$	0.8	42.2	5.08	0.2167	18.06469	42.875
$Mg(NO_3)_2$	1.5	39.57	3.29	0.2226	18.06731	42.702
$MgCl_2$	0.5	15.8	5.88	0.3326	18.05916	41.636
$Co(NO_3)_2$	0.5	28.7	13.13	0.1108	18.06502	43.688
$NiCl_2$	0.5	16.42	−6.58	0.2281	18.07590	42.500
$CoCl_2$	0.5	13.9	2.92	0.1821	18.06734	42.922
$(NH_4)_2SO_4$	0.8	53.35	6.61	0.2198	18.06307	42.469
$CuSO_4$	0.5	−4.9	9.72	0.2193	18.06039	42.344
$AlCl_3$	0.5	12.7	9.76	0.2565	18.05790	41.595
$MgSO_4$	0.7	−1.7	8.29	0.1725	18.06413	42.619
$K_2Cr_2O_7$	1.0	92.45	11.52	0.1089	18.06692	43.368
Na_2CO_3	0.7	−3.9	9.78	0.5618	18.02279	37.531
$Al(NO_3)_3$	0.5	47.30	4.93	0.1076	18.06783	43.268
K_2SO_4	1.0	27.4	6.09	0.2723	18.06118	41.798
$Co(C_2H_3O_2)_2$	1.0	34.0	101.1	0.09681	18.05222	43.390
K_2CO_3	0.8	14.9	8.55	0.2135	18.06170	41.757
$Ba(OH)_2$	1.0	2.19	−37.73	0.1737	18.08996	42.290
H_2O					18.06941	44.718

[25] J. D. Bernal and R. H. Fowler, J. Chem. Phys. **1**, 515 (1933).
[26] L. Pauling, *The Nature of the Chemical Bond* (Cornell University Press, 1940), pp. 47, 348.
[27] H. S. Frank and A. L. Robinson, J. Chem. Phys. **8**, 933 (1940).
[28] W. Geffcken, Zeits. f. physik. Chemie **167A**, 240 (1933).
[29] H. E. Wirth, J. Am. Chem. Soc. **62**, 1128 (1940).
[30] G. W. Stewart, J. Chem. Phys. **7**, 381 (1939).

over a great concentration range. With x so chosen, the equation $\Phi_2=\Phi_2^0+kc_2^x$ is linear, with slope k and intercept Φ_2^0.

Since experimentally $\Phi_2=\Phi_2^0+kc_2^x$,

$$\partial\Phi_2/\partial c_2=kxc_2^{x-1},$$

$$c_2=1000n_2/V,$$

$$\partial c_2/\partial n_2=1000\left\{\frac{V-n_2(\partial V/\partial n_2)}{V^2}\right\}.$$

From (5) with substitutions:

$$\frac{\partial V}{\partial n_2}=\bar{V}_2=\frac{\Phi_2^0+k(x+1)c_2^x}{1+(kxc_2^{x+1}/1000)}. \qquad (6)$$

As one can easily verify from columns 3, 5, and 6 of Table III, the second term in the denominator of (6) is negligible, especially in comparison with errors in empirical determination of Φ_2^0, k, and x. Therefore with this approximation it results that

$$\bar{V}_2=\Phi_2^0+k(x+1)c_2^x. \qquad (7)$$

That $\Phi_2^0=\bar{V}_2^0$ is of course apparent from definitions of Φ_2 and $\bar{V}_2$.

Substituting (7) into (3b) there results:

$$\bar{V}_1=\bar{V}_1^0\frac{1000-c_2\Phi_2^0-k(x+1)c_2^{x+1}}{1000-c_2\Phi_2^0-kc_2^{x+1}}. \qquad (8)$$

When Φ_2 varies linearly with $c^{\frac{1}{2}}$ (i.e., $x=\frac{1}{2}$), (8) reduces to Wirth's formula for $\bar{V}_1$.

Equation (4) which defines $\Phi(\bar{V}_2)$ is readily transformed to an expression suitable for computation. Introducing the density of the solution ρ, the molecular weight of the solute M_2, and the density of the solvent ρ_1, it follows that:

$$\Phi(\bar{V}_2)=\Phi_2=\frac{M_2}{\rho_1}-\frac{1000}{\rho_1}\frac{(\rho-\rho_1)}{c_2};$$

$c_2=10P\rho/M_2$ (P = percent by weight of solute).

Table III was prepared on the basis of the preceding development. From information given in the *International Critical Tables*, Vol. III, supplemented in a few cases by density-temperature data from this research, values of Φ_2 and corresponding values of c_2 were computed at 25°C. For each electrolyte a sufficient number of plots of Φ_2 against powers of c_2 were made to permit the selection of the linear graph. Slope, intercept, and exponent as determined in this manner appear directly in Table III, and as well were used together with proper values of c_2 in finding $\bar{V}_1$ from (8). It is true that the values of Φ_2^0 appearing in Table III have been obtained by extrapolation and are not necessarily the values which Φ_2 would assume at zero concentration. However, a comparison of (3b) and (8) reveals that Φ_2^0 and $\bar{V}_2^0$ appear in (8) only as part of Φ_2 and $\bar{V}_2$, respectively; and that consequently, linearity of the plot of Φ_2 against c^x in the concentration range containing the measurement is the sole requirement for the applicability of (8). The interpolation factor of (8) has virtually the same value whether the cubic centimeter or the milliliter is used as the volume unit. Thus the volume unit in terms of which $\bar{V}_1$ is expressed in Table III depends upon the fact that $\bar{V}_1^0$ is expressed in cubic centimeters.

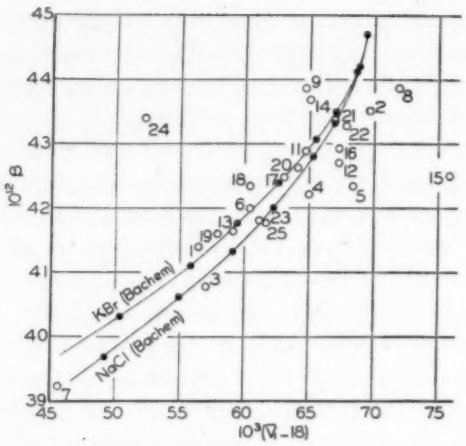

FIG. 3. Correlation between adiabatic compressibility of solutions and partial molal volume of solvent water at 25°C for the following numbered electrolytes: 1. NH_4Cl, 2. $LiNO_3$, 3. LiCl, 4. $NH_4C_2H_3O_2$, 5. KCN, 6. KCl, 7. NaCl, 8. $KClO_3$, 9. $CoSO_4$, 10. NaOH, 11. $Ca(NO_3)_2$, 12. $Mg(NO_3)_2$, 13. $MgCl_2$, 14. $Co(NO_3)_2$, 15. $NiCl_2$, 16. $CoCl_2$, 17. $(NH_4)_2SO_4$, 18. $CuSO_4$, 19. $AlCl_3$, 20. $MgSO_4$, 21. $K_2Cr_2O_7$, 22. $Al(NO_3)_3$, 23. K_2SO_4, 24. $Co(C_2H_3O_2)_2$, 25. K_2CO_3, 26. $Ba(OH)_2$.

The circles of Fig. 3 represent experimental information on the electrolytes of this research, each at a single concentration (Tables I, III). The dots defining the two curves represent values of β at solute concentrations of 0, 0.1, 0.25, 0.35, 0.5, 0.65, 0.8, and 1.0 mole per liter, respectively, found by graphical interpolation from Bachem's

work,[31] and corresponding values of $\bar{V}_1$ computed therefrom by the author in the manner already described. In most cases, irrespective of the type of solute, there is a surprising correlation between corresponding values of β and $\bar{V}_1$.

The significance of the correlation is more readily appreciated through physical interpretation. In nearly all cases the addition of an electrolyte produces two intuitively compatible results: the partial molal volume of the solvent decreases, and the compressibility of the solution becomes less. These are the results one would anticipate if the water structure were more highly coordinated by the addition of ions. On the other hand, if compressibility values increased in general with decreasing partial molal volume of the solvent, one could not confidently maintain that a more highly coordinated water structure is responsible for the observed change.

That the two full curves from Bachem diverge slightly is not at all strange when one considers the effects which such factors as ionic radius, configuration, associated ion pairs, etc., might have upon both β and $\bar{V}_1$. More significant is the surprisingly close correlation which is shown between these two curves, and by the general clustering of the circles for nearly all of the electrolytes investigated. One is led to the conclusion that in most cases, solvent structure change exercises an unmistakable influence upon both β and $\bar{V}_1$.

According to the *International Critical Tables* from which values of Φ_2 are computed, $KClO_3$, $NiCl_2$, and $Ba(OH)_2$ are peculiar in that their apparent molal ionic volumes decrease with concentration. The slope of the curve for $NiCl_2$ changes, and even reverses in sign as the temperature rises from 17°C to 25°C. These cases, apparently anomalous, are represented on Fig. 3 by circles 8, 15, and 26.[32] Curves belonging to these electrolytes would thus show an increasing β with decreasing $\bar{V}_1$, in contrast with all other electrolytes studied. Recent measurements by Redlich and Bigeleisen,[33] however, have shown the Critical Tables to be in error with regard to the anomalous character of $KClO_3$. It is possible, therefore, that the Tables are erroneous in the information which predicts anomalous behavior of $NiCl_2$ and $Ba(OH)_2$.

A perfect correlation between β and $\bar{V}_1$ is hardly to be expected since there are a number of contributing factors in each case and since computation of $\bar{V}_1$ involves an empirical evaluation of constants. The reason is not obviously indicated for the peculiar position of $Co(C_2H_3O_2)_2$, which seems to have more effect upon solvent structure than upon solution compressibility (cf. Fig. 2); or for the possibly anomalous behavior of $NiCl_2$ and $Ba(OH)_2$, for which the curves of Φ_2 against c^x have negative slopes at 25°C. In such unusual cases there are clearly other agencies besides the compacted solvent structure which account for the observed effects.

From a positive standpoint, however, the results are definite and heartening. For all electrolytes the mole fractional change in compressibility is satisfactorily correlated with x-ray and other indications of liquid structure change in the water; and the compressibilities are themselves correlated in a striking manner with corresponding values of the partial molal volume of the solvent water. All such facts strongly support the conclusion that the water structure becomes more highly coordinated and compacted with the introduction of ions. No well-substantiated evidence presented in this paper maintains a contrary view. Further research has been instituted on some of the problems which have arisen during the prosecution of this research.

[31] See reference 5.
[32] Circle 26 is outside graphed region.
[33] Redlich and Bigeleisen, J. Am. Chem. Soc. **64**, 758 (1942).

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Energy Eigenvalues for the Coulomb Potential with Cut-Off. Part I.

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In this paper, the Schrödinger equation is solved analytically for the Coulomb potential with horizontal cut-off. For this purpose, a detailed mathematical study of the confluent hypergeometric functions is carried out and their similarity to Bessel and Neumann functions is put on a quantitative basis. The resulting formulas for the energy levels are valid for all cases except when the energy is lying very little above the cut-off.

THE Coulomb potential with a cut-off (see Fig. 1) has been proposed as an approximate potential energy for certain quantum mechanical problems in the theory of solids.[1] It has actually been used for such a purpose by Tibbs, who solved the problem in one particular case by numerical integration methods.[2] But, in fact, the problem is capable of an analytic, though not rigorous, solution which is to be derived in the following.

The problem is to solve the equation

$$\frac{d^2F}{dr^2}+\left(-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r_0}-\frac{l(l+1)}{r^2}\right)F=0 \qquad (1)$$

for $r<r_0$ and

$$\frac{d^2F}{dr^2}+\left(-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r}-\frac{l(l+1)}{r^2}\right)F=0 \qquad (2)$$

for $r>r_0$, and to join them with continuous derivative at $r=r_0$. The usual conditions are to be added at the origin and infinity. The energy parameter is written here in the form $-\alpha^2/k^2$, where α is a given constant and k is our unknown to be determined.

1. BESSEL-LIKE AND NEUMANN-LIKE FUNCTIONS

The solutions of Eq. (2) are called confluent hypergeometric functions and are discussed in Whittaker and Watson.[3] The two solutions may be either distinguished by their behavior at the origin or their behavior at infinity. In the notation of WW, we have one solution, $M_{k,l+\frac{1}{2}}(2\alpha r/k)$, which is finite at the origin while the other, $M_{k,-l-\frac{1}{2}}(2\alpha r/k)$, has a pole of order l. At plus infinity, one solution, $W_{k,l+\frac{1}{2}}(2\alpha r/k)$, is finite while the other tends to infinity. It is well known that whenever the expression $k-l$ is a positive integer, the solution which is finite at the origin also happens to be finite at $+\infty$. Hence $k=l+1, l+2, l+3, \cdots$ are the eigenvalues of k in the case of the Coulomb potential without cut-off. If there is a cut-off at r_0, then $W_{k,l+\frac{1}{2}}(z)$ is the correct wave function beyond r_0, but the values of k will no longer be integer.

The similarity between the functions M and the Bessel functions is well known.[4] In order to bring it out we introduce a new variable x which is defined thus:

$$x=2(2\alpha r)^{\frac{1}{2}}=2(kz)^{\frac{1}{2}} \qquad (3)$$

and instead of M we introduce $J_p^k(x)$,

$$J_p^k(x)=\frac{z^{-\frac{1}{2}k^{\frac{1}{2}}p}}{p!}M_{k,\frac{1}{2}p}(z). \qquad (4)$$

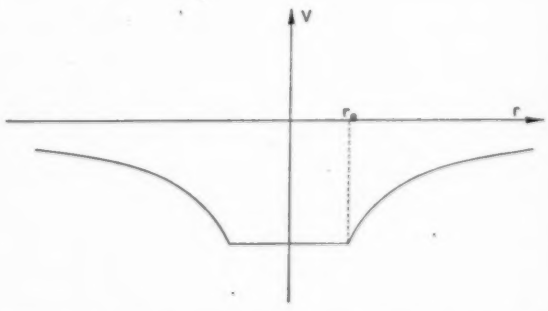

FIG. 1. Coulomb potential with cut-off. Solving the Schrödinger equation for such a potential is the problem of this paper.

[1] G. H. Wannier, Phys. Rev. **52**, 191 (1937).
[2] S. R. Tibbs, Trans. Faraday Soc. **35**, 1471 (1939).
[3] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press), Chapter XVI. In the following quoted as WW.
[4] H. A. Bethe, *Handbuch der Physik*, second edition, Vol. XXIV, 1, p. 287.

It can be verified by substitution into Eq. (2) that

$$\lim_{k\to\infty} J_p^k(x)=J_p(x),$$

i.e., the zero energy Schroedinger eigenfunction is a Bessel function. In addition, by comparing power series, one finds that the first two terms of the series agree for all values of k. A better standard of comparison is gained by a study of their integral representations. The well-known Bessel integral may take the following form

$$J_p(x)=\frac{1}{2\pi i}\int \exp\left[\tfrac{1}{2}x\left(u-\frac{1}{u}\right)\right]u^{-p-1}du. \quad (5)$$

The path comes from $-\infty$, circles around the origin in a positive direction and returns to $-\infty$ (Fig. 2). A similar integral exists for $J_p^k(x)$. Whittaker and Watson introduce an integral representation for $W_{k,m}$ only.[5] The same integral will also be valid for $M_{k,m}$ if the path is suitably altered. Transforming their integral with (3) and (4) and introducing a new variable of integration u through

$$t=-\tfrac{1}{2}(xu-z)=-\tfrac{1}{2}x[u+(x/4k)],$$

we get

$$J_p^k(x)=\frac{1}{2\pi i}\int e^{\frac{1}{2}xu}\left(u+\frac{x}{4k}\right)^{-k-\frac{1}{2}(p+1)}\times\left(u-\frac{x}{4k}\right)^{k-\frac{1}{2}(p+1)}\text{du.} \quad (6a)$$

The path in WW must be altered to encircle both singular points of the integrand in a counterclockwise direction and then to return to $-\infty$ (Fig. 3). This statement is verified as

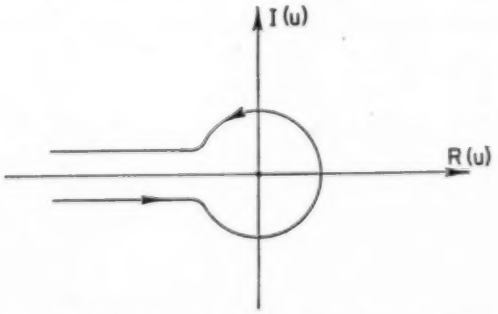

FIG. 2. Path of integration to be followed in the definition of the Bessel function by the Bessel integral [Eq. (5)].

[5] WW, section 16·12.

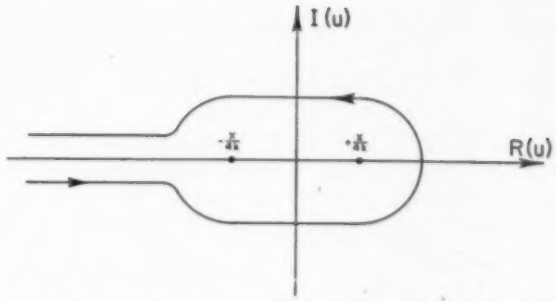

FIG. 3. Path of integration to be followed in defining the Bessel-like function by a contour integral [Eq. (6)].

follows: If we substitute t back again into (6) we get

$$J_p^k(x)=-\exp\left(-\frac{x^2}{8k}\right)(\tfrac{1}{2}x)^p(1/2\pi i)\times\int e^{-t}(-t)^{-p-1}[1+(x^2/4kt)]^{k-\frac{1}{2}(p+1)}dt.$$

Since the path encircles both singularities it can be deformed to make $|t|>x^2/4k$. Hence the binomial can be developed into a power series, and the series can be integrated term by term. The resulting series, by (3) and (4), agrees with the series given for M in WW.

To bring out the similarity between (5) and (6) we write the latter in the form

$$J_p^k(x)=(1/2\pi i)\times\int \exp\left[\tfrac{1}{2}x\left(u-\frac{1}{u}\right)\right]u^{-p-1}\phi(x/4ku)du, \quad (6b)$$

with

$$\phi(s)=(1-s^2)^{-\frac{1}{2}(p+1)}\left[e^{2s}\frac{1-s}{1+s}\right]^k \quad (7)$$
$$=1+\tfrac{1}{2}(p+1)s^2-\tfrac{2}{3}ks^3+\cdots.$$

Since $|u|$ may be made larger than 1 in the integrand, we can conclude that

$$J_p^k(x)=J_p(x)\left[1+0\left(\frac{x^2}{16k^2}\right)\right]. \quad (8)$$

It is possible, incidentally, to derive a development of the Bessel-like function $J_p^k(x)$ in a series of Bessel functions, following up the reasoning just indicated. It can be stated as follows: If we write for the series (7)

$$\phi(s)=\sum_{\nu=0}^{\infty} b_\nu s^\nu, \quad (9a)$$

then

$$J_p^k(x)=\sum_{\nu=0}^{\infty} b_\nu (x/4k)^\nu J_{p+\nu}(x). \qquad (9b)$$

The convergence is quite rapid, but there are other, more physical ways of constructing improved expressions for $J_p^k(x)$ which are outlined in Sections 3 and 4 below.

In working with Bessel-like functions of integer index p we are faced with a difficulty which is well known from the theory of Bessel functions. Usually, J_p^k and J_{-p}^k form a complete solution of our differential Eq. (2), but whenever p becomes integer the two solutions become identical, apart from a constant factor. We must therefore define an auxiliary function to use beside J_p^k, and it is natural to look for a function which may be compared with the Neumann function $N_p(x)$ (called $Y_p(x)$ in some books). In the latter case, the definition runs

$$N_p(x)\sin p\pi = J_p(x)\cos p\pi - J_{-p}(x). \qquad (10)$$

The right-hand side of this equation is constructed so as to vanish identically for integer p. The same principle must be applied in constructing a Neumann-like function N_p^k from the two Bessel-like functions J_p^k and J_{-p}^k. The relationship between the latter two may be obtained from their power series. If we add to this relationship two requirements which we wish to impose, namely,

$$\lim_{x=0}\frac{N_p^k(x)}{N_p(x)}=1 \quad \text{for all } k$$

and

$$\lim_{k\to\infty}\frac{N_p^k(x)}{N_p(x)}=1 \quad \text{for all } x,$$

then the new function is completely determined:

$$N_p^k(x)\sin p\pi = \frac{(k+\frac{1}{2}p-\frac{1}{2})!}{(k-\frac{1}{2}p-\frac{1}{2})!k^p}J_p^k(x)\cos p\pi - J_{-p}^k(x). \qquad (11)$$

We can expect the new function to equal $N_p(x)$ under the same condition (8) as in the previous case.[6]

[6] It appears at first sight that the factor in front of J_p^k alters the situation because it is possible to satisfy the requirement (8)

$$x\ll 4k$$

and yet to have that factor markedly different from 1 by having

$$p\sim 2k.$$

But these two conditions imply also

$$x<p$$

and under these conditions, the lowest negative power in J_{-p}^k outweighs all other contributions to the function N.

The Bessel-like and the Neumann-like function form a complete system of solutions for (2), and therefore, our wave function $W_{k,m}(z)$ is expressible in terms of them. These relationships are well known.[7] We may write the relation relevant to our case

$$z^{-\frac{1}{2}}W_{k,m}(z)\sin 2m\pi = (k+m-\tfrac{1}{2})!k^{-m}J_{2m}^k(x)\sin(k+m-\tfrac{1}{2})\pi - (k-m-\tfrac{1}{2})!k^m J_{-2m}^k(x)\sin(k-m-\tfrac{1}{2})\pi. \qquad (12)$$

This relation reduces to $0=0$ for integer or half integer m. But if we eliminate J_{-2m}^k with the help of (11) we get a non-trivial relation for all m

$$z^{-\frac{1}{2}}W_{k,m}(z) = (k+m-\tfrac{1}{2})!k^{-m}J_{2m}^k(x)\cos(k-m-\tfrac{1}{2})\pi - (k-m-\tfrac{1}{2})!k^m N_{2m}^k(x)\sin(k-m-\tfrac{1}{2})\pi. \qquad (13)$$

Through its trigonometric factors, the formula shows up clearly the oscillating character of W as a function of k. Every time $k-m+\frac{1}{2}$ is a positive integer the term in N will drop out. The oscillatory character does not continue, however, into the region where $k<m$ because the vanishing of the sine is cancelled by the pole in the factorial.

2. A SIMPLE APPROXIMATE SOLUTION OF THE PROBLEM

Our problem consists in fitting the solution of (1), namely,

$$F_1 = xJ_{l+\frac{1}{2}}\left(\frac{x^2}{2x_0}\left(1-\frac{x_0^2}{16k^2}\right)^{\frac{1}{2}}\right), \qquad (14)$$

where we have defined consistently with (3)

$$x_0=2(2\alpha r_0)^{\frac{1}{2}} \qquad (15)$$

and the solution of (2), which may be written according to (13),

$$F_2 = xJ_{2l+1}^k(x)\cos k\pi + \frac{(k-l-1)!k^{2l+1}}{(k+l)!}xN_{2l+1}^k(x)\sin k\pi. \qquad (16)$$

[7] WW, section 16·41.

It is possible to get a quick and simple answer by observing the following. We have a condition under which the Bessel-like and Neumann-like function can be replaced by their namesakes, namely,

$$x \ll 4k. \tag{17}$$

Now it is very fortunate that for physical reasons a somewhat weaker condition is always satisfied at the cut-off $x=x_0$:

$$x < 4k. \tag{18}$$

We obtain the condition by demanding that the total energy be higher than the potential energy at the cut-off. Condition (17) will, in fact, apply to all but some of the lowest levels although the latter are no doubt the most important ones. Keeping this restriction in mind we shall construct an approximate solution valid for most states by setting equal at $x=x_0$ the logarithmic derivatives of the following functions

$$F_1 = xJ_{l+\frac{1}{2}}(x^2/2x_0) \tag{19}$$

and

$$F_2 = xJ_{2l+\frac{1}{2}}(x)\cos k\pi + \frac{(k-l-1)!k^{2l+1}}{(k+l)!} xN_{2l+\frac{1}{2}}(x)\sin k\pi. \tag{20}$$

We find the following result

$$\frac{(k-l-1)!k^{2l+1}}{(k+l)!}\tan k\pi = -\frac{J_{2l}(x)J_{l+\frac{1}{2}}(\frac{1}{2}x) - J_{2l+\frac{1}{2}}(x)J_{l-\frac{1}{2}}(\frac{1}{2}x)}{N_{2l}(x)J_{l+\frac{1}{2}}(\frac{1}{2}x) - N_{2l+\frac{1}{2}}(x)J_{l-\frac{1}{2}}(\frac{1}{2}x)}, \tag{21}$$

where we have replaced for brevity the cut-off coordinate x_0 by x. Equation (21) is explicit in that it contains on the right a function of the cut-off coordinate only and on the left a function of the unknown k, the "effective" principal quantum number. For s states the formula is particularly attractive:

$$\tan k\pi = -\frac{J_0(x)\sin\frac{1}{2}x - J_1(x)\cos\frac{1}{2}x}{N_0(x)\sin\frac{1}{2}x - N_1(x)\cos\frac{1}{2}x}. \tag{22}$$

To realize the power of this simple formula we shall recompute with its help the results of Tibbs.[2] Tibbs calculated in a particular case the energy values for the $1s$, $2s$, and $2p$ states. His cut-off is located at

$$x = 2.62$$

and he found for k:

$$k = 1.154 \pm .002 \quad \text{in the } 1s \text{ state,}$$
$$k = 2.143 \pm .019 \quad \text{in the } 2s \text{ state.}$$

Formula (22) gives

$$k = n + 0.158 \quad \text{for all } s \text{ states.}$$

The agreement may be partly accidental because it is upset if a refined approximation is used for one of the wave functions only. However, it is not impossible that formula (22) is actually better than it appears from this paper and that the deduction of the author is not doing it full justice. In any case, formulas (21) and (22) are very useful for a rough location of the energy levels.

What follows first from (21) and (22) is that while the energy correction is of course smaller for high energy levels, the correction to the principal quantum number is of the same order for all levels having the same angular momentum quantum number. It is actually the same (within the restrictions of this section) for all s states and tends to be the same for sufficiently high energy in the other cases. If we turn to comparing states of different l, then s states are more disturbed than p states, and so forth. One can appreciate that difference particularly well if one assumes x sufficiently small to permit power series development of the Bessel and Neumann functions. We get in that case from (21)

$$k = n + \left(1-\frac{1}{n^2}\right)\left(1-\frac{4}{n^2}\right)\cdots\left(1-\frac{l^2}{n^2}\right) \times \frac{(\frac{1}{2}x)^{4l+4}}{(2l+1)!(2l+3)!},$$

where n is the next lower integer, i.e., the principal quantum number in the correct sense of the word. The limiting formula for large x is also very simple if the Bessel and Neumann functions can be replaced by their trigonometric equivalents and if the ratio of factorials can be set equal to 1. We get

$$k = n + (x/2\pi) - \tfrac{1}{2}l - \tfrac{1}{4}.$$

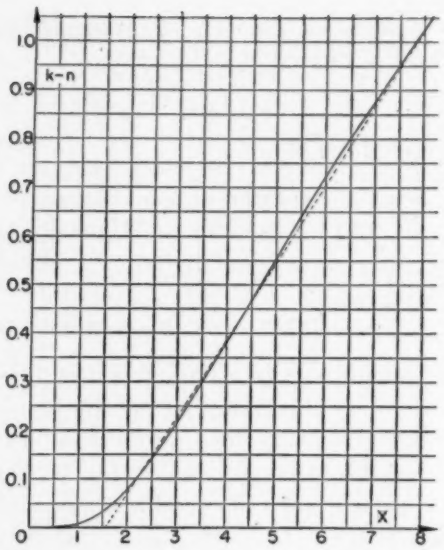

FIG. 4. Plotting energy against cut-off radius according to Eq. (22). The abscissa x goes as the square root of the radius [Eq. (3)] and the ordinate contains the difference of the principal quantum number n and a number k which must be substituted into the Balmer formula to give the energy levels. The graph applies only to s states under the restriction (17).

The validity of this formula is somewhat restricted because x must be large compared to l, yet condition (17) must not be broken. A graphical evaluation of formula (22), plotting $k-n$ against x for s states is given in Fig. 4. One sees that the curve oscillates about the asymptote which we discussed just above.

3. A MORE PRECISE SOLUTION FOR s STATES

It has been pointed out that the solution of Section 2 is correct if condition (17) holds while only condition (18) is fulfilled in many actual cases. In this section, we shall improve Eq. (22) by applying various corrections to the approximation (20), so as to make it quantitatively correct everywhere except in the immediate neighborhood of $x=4k$.

When one studies the approximate wave functions (20) numerically and compares them with the Schrödinger eigenfunctions in the case of integer k, one finds of course complete disagreement beyond $x=4k$. Below that point the two functions show the same general structure, but vary somewhat in detail. It is possible to cancel out most of this discrepancy by the following two steps.

In the first step, we carry out a change in the abscissa scale so as to bring the nodes of the true wave function and its approximation to coincidence; i.e., we write the wave function in the form

$$F=x_kZ_1(x_k),$$

where $Z_1(x)$ stands for the linear combination (20) of Bessel and Neumann functions. x_k is an unknown function of x differing from it very little as long as condition (17) holds. If we introduce a corresponding r_k through (3) we get for F the equation

$$\left(\frac{d^2F}{dr_k^2}\right)+\left(\frac{2\alpha}{r_k}\right)F=0,$$

while as a function of r, it obeys Eq. (2),

$$\frac{d^2F}{dr^2}+\left(-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r}\right)F=0.$$

These two equations give for r_k as a function of r

$$-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r}=\frac{2\alpha}{r_k}\left(\frac{dr_k}{dr}\right)^2-\frac{1}{F}\frac{dF}{dr_k}\frac{d^2r_k}{dr^2}.$$

It is approximately correct to leave off the second term on the right-hand side because $(1/F)(dF/dr_k)$ is an expression of alternating sign which will make little long range contribution to $(dr_k/dr)^2$. The remainder of the equation reads then

$$\frac{dr_k}{dr}=r_k^{\frac{1}{2}}\left(\frac{1}{r}-\frac{\alpha}{2k^2}\right)^{\frac{1}{2}}$$

or passing over to x coordinates with the help of (3)

$$\frac{dx_k}{dx}=\left(1-\frac{x^2}{16k^2}\right)^{\frac{1}{2}}$$

which integrates to

$$x_k=\tfrac{1}{2}x\left(1-\frac{x^2}{16k^2}\right)^{\frac{1}{2}}+2k\arcsin\frac{x}{4k}. \qquad (23)$$

TABLE I. Approximations of the $2s$ eigenfunction.

$2\alpha r$	x	$F_{(20)}$	$F_{(23)}$	F
0	0.000	0.000	0.000	0.000
1	2.000	1.154	1.168	1.168
4	4.000	−0.264	0.010	0.000
6	4.899	−1.540	−1.308	−1.337
8	5.657	−1.861	−2.181	−2.164
9	6.000	−1.660	−2.414	−2.370
12	6.928	−0.180	−2.642	−2.392
16	8.000	1.876	$-\infty$	−1.758

The behavior of x_k/k as a function of x/k is shown in Fig. 5. At first, they are very similar, but as the latter approaches 4, the former falls back and approaches π with zero slope.

This correction brings the nodes quite well to coincidence. One finds now the main discrepancy in the size of the loops, the true loops increasing somewhat more sharply with r than the approximate ones. To correct this, we introduce as our second step an unknown factor f into the wave function, writing

$$F=f\cdot\Phi,$$

where

$$\Phi=x_kZ_1(x_k).$$

It is clear from the purpose of the correction that f may be assumed to vary slowly compared to Φ. Substituting into (2) and using the equation for Φ, *viz.*,

$$\frac{d^2\Phi}{dr^2}-\frac{d^2r_k/dr^2}{dr_k/dr}\frac{d\Phi}{dr}+\left(-\frac{\alpha^2}{k^2}+\frac{2\alpha}{r}\right)\Phi=0,$$

we find

$$\frac{d^2f}{dr^2}\Phi+\left\{2\frac{df}{dr}+f\frac{d^2r_k/dr^2}{dr_k/dr}\right\}\frac{d\Phi}{dr}=0.$$

According to our assumption, the first term can be neglected compared to the others and we get

$$f=\left(\frac{dr_k}{dr}\right)^{-\frac{1}{2}}=\frac{x}{x_k^{\frac{1}{2}}x_k^{*\frac{1}{2}}}, \tag{24}$$

where we have introduced the abbreviation

$$x_k^*=\frac{dx_k}{dx}x=x\left(1-\frac{x^2}{16k^2}\right)^{\frac{1}{2}}. \tag{25}$$

Thus we get finally for F

$$F=xx_k^{\frac{1}{2}}x_k^{*-\frac{1}{2}}\{J_1(x_k)\cos k\pi+N_1(x_k)\sin k\pi\}. \tag{26}$$

It is perhaps worth while to explain the corrections of this section by physical reasoning. The function

$$xZ_1(x)$$

is the correct solution of (2) for zero energy. According to the Wentzel-Kramers-Brillouin approximation, the true wave function for a negative energy state must differ from it on two counts:

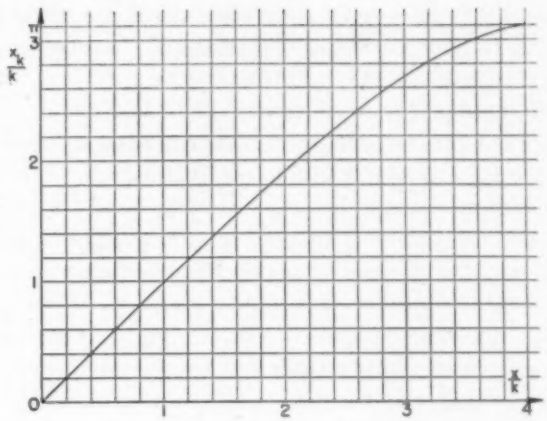

FIG. 5. Plotting the scale-corrected argument x_k against the true x [Eq. (23)]. x_k must be used instead of x as the argument in the Bessel or Neumann function to make our approximation quantitatively correct. It is an essential feature of the method that it does not extend beyond $x=4k$.

(a) The wave-length, obeying De Broglie's relation, must be longer. This reasoning leads to a change in the abscissa scale which agrees exactly with (23).

(b) The amplitude of the wave function is inversely proportional to the square root of the classical velocity. Hence we should multiply our wave function with a factor f which equals

$$f=\left(\frac{\text{false velocity at false position}}{\text{right velocity at right position}}\right)^{\frac{1}{2}}.$$

This formula reproduces exactly (24).

It is an inevitable consequence of such reasoning that the corrected wave function breaks down at $x=4k$ because this is the limit of the classical orbit. But until the neighborhood of that point is reached, the approximation is greatly improved numerically. To illustrate this fact, the Schrödinger wave function for the $2s$ state is given in Table I, and the approximations (20) and (26) are compared with it.

Fitting the wave function (26) to the wave function (15) at the point $x=x_0$ is now a straightforward matter. One finds

$$\tan k\pi=-\frac{J_0(x_k)\sin\frac{1}{2}x_k^*-J_1(x_k)\cos\frac{1}{2}x_k^*+\frac{1}{2}\{(x^2/x_k^{*3})-(1/x_k)\}J_1(x_k)\sin\frac{1}{2}x_k^*}{N_0(x_k)\sin\frac{1}{2}x_k^*-N_1(x_k)\cos\frac{1}{2}x_k^*+\frac{1}{2}\{(x^2/x_k^{*3})-(1/x_k)\}N_1(x_k)\sin\frac{1}{2}x_k^*}. \tag{27}$$

x_k and x_k^* are derived from the cut-off coordinate $x=x_0$ by the formulas (23) and (25). x_0 itself is derived from r_0 through (15).

Formula (27) is of course less convenient to use than (22) because the unknown k is contained implicitly in the right-hand expression. That expression varies, however, much more slowly with k than the left (except in the excluded case when x is very close to $4k$) and hence the formula is suitable for successive approximations, starting with an approximate k on the right. One or two steps are amply sufficient to get the answer.

Applying the formula to the case of Tibbs[2] we get

$k=1.153$ for the $1s$ state
$k=2.159$ for the $2s$ state

in complete agreement with his results.

4. SOLUTION FOR OTHER STATES

The method of Section 3 is applicable in the same form to states of non-vanishing angular momentum. We shall restrict ourselves therefore to the discussion of new features and the quotation of results.

Following the same reasoning as previously we are led to an equation for the scale-corrected radius r_{kl}

$$\left(\frac{dr_{kl}}{dr}\right)^2=\frac{\dfrac{2\alpha}{r_{kl}}-\dfrac{l(l+1)}{r_{kl}^2}}{\dfrac{2\alpha}{r}-\dfrac{l(l+1)}{r^2}-\dfrac{\alpha^2}{k^2}}.$$

We notice that this expression is considerably more involved than in the previous case. This is explained with the help of Fig. 6, which shows the potential for radial motion. In the actual negative energy state the classical orbit has now an inner limit b as well as an outer limit c which are the two roots of the denominator above; and the zero-energy orbit has an inner limit a, which is the root of the numerator. It follows from this that r_{kl} will be a complex function of r unless the constant of integration is picked in such a way that when

$$r=b$$

then

$$r_{kl}=a.$$

Such a choice has the additional advantage that it brings the first inflexion points of the approximate and the true F automatically to coincidence. We find thus

$$(r_{kl}-a)^{\frac{1}{2}}-a^{\frac{1}{2}}\arctan\left(\frac{r_{kl}-a}{a}\right)^{\frac{1}{2}}=\tfrac{1}{2}\left(\frac{(r-b)(c-r)}{b+c}\right)^{\frac{1}{2}}+\tfrac{1}{2}(b+c)^{\frac{1}{2}}\arctan\left(\frac{r-b}{c-r}\right)^{\frac{1}{2}}-a^{\frac{1}{2}}\arctan\left(\frac{\frac{1}{b}-\frac{1}{r}}{\frac{1}{r}-\frac{1}{c}}\right)^{\frac{1}{2}}. \quad (28)$$

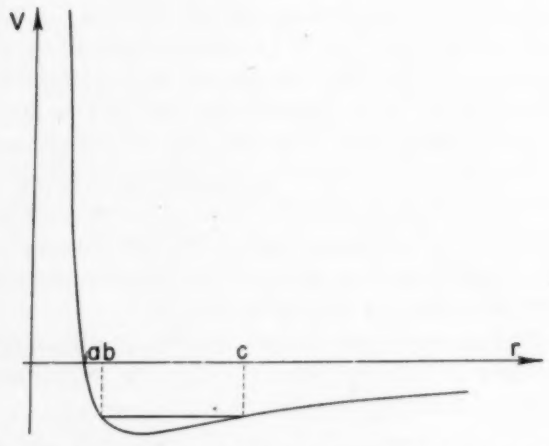

FIG. 6. Radial potential for a particle having non-vanishing angular momentum which moves under the influence of a Coulomb force.

The new formula is seen to go over into (23) in the special case when $a=b=0$.

No special difficulty arises in the determination of the factor f. If we define as in (3)

$$x_{kl}=2(2\alpha r_{kl})^{\frac{1}{2}} \quad (29)$$

and as in (25)

$$x_{kl}^*=\frac{dx_{kl}}{dx}x=x\left[\frac{1-\dfrac{4l(l+1)}{x^2}-\dfrac{x^2}{16k^2}}{1-\dfrac{4l(l+1)}{x_k^2}}\right]^{\frac{1}{2}}, \quad (30)$$

then we get for the wave function F:

$$F=xx_{kl}^{-\frac{1}{2}}x_{kl}^{*-\frac{1}{2}}Z_{2l+1}(x_{kl}),$$

where $Z_{2l+1}(x)$ is a combination of Bessel and Neumann functions of order $2l+1$. It seems logical to follow the procedure that led to (26), that is, to substitute for Z the linear combination (20). However, such a procedure is incorrect because the substitution of x_{kl} for x has brought in some constant factors. x_{kl}, through (28), (29), and (3), is a regular function of x right through the inner limit point of the orbit $x_{kl}=2[l(l+1)]^{\frac{1}{2}}$; the same applies to x^*_{kl} as defined by (30). They both are also equal to zero simultaneously with x, and their ratio x^*_{kl}/x_{kl} tends to 1 in that limit; but the ratio x_{kl}/x does not do so. We find

$$\lim_{x=0}\frac{x_{kl}}{x}=\left[\frac{\left(1+\dfrac{[l(l+1)]^{\frac{1}{2}}}{k}\right)^{\{k/[l(l+1)]^{\frac{1}{2}}\}+1}}{\left(1-\dfrac{[l(l+1)]^{\frac{1}{2}}}{k}\right)^{\{k/[l(l+1)]^{\frac{1}{2}}\}-1}\cdot e^2}\right]^{\frac{1}{2}}$$

which is a quantity lying between 1 and 0.8578. Now the Bessel-like and Neumann-like functions are characterized by their behavior at the origin where they are to behave as

$$\frac{(\frac{1}{2}x)^{2l+1}}{(2l+1)!}\quad\text{and}\quad-\frac{(2l)!(\frac{1}{2}x)^{-2l-1}}{\pi},$$

respectively. Hence, if we define

$$C_{kl}=\left[\frac{\left(1-\dfrac{[l(l+1)]^{\frac{1}{2}}}{k}\right)^{\{k/[l(l+1)]^{\frac{1}{2}}\}-1}\cdot e^2}{\left(1+\dfrac{[l(l+1)]^{\frac{1}{2}}}{k}\right)^{\{k/[l(l+1)]^{\frac{1}{2}}\}+1}}\right]^{l+\frac{1}{2}},\qquad(31)$$

we get F equal to

$$F=xx_{kl}^{-\frac{1}{2}}x_{kl}^{*-\frac{1}{2}}\left\{C_{kl}^{\frac{1}{2}}J_{2l+1}(x_{kl})\cos k\pi+C_{kl}^{-\frac{1}{2}}\frac{(k-l-1)!k^{2l+1}}{(k+l)!}N_{2l+1}(x_{kl})\sin k\pi\right\}.\qquad(32a)$$

TABLE II. Comparison of the 4d wave function with approximation (32a).

x^2	x^2_{kl}	$F_{(32a)}$	F
0.0	0.0	0.000	0.000
26.8	24.0	0.782	0.782
50.0	43.2	1.639	1.632
80.0	65.6	0.940	0.913
120.0	91.5	−1.295	−1.323
160.0	112.5	−2.53	−2.39
229.2	132.6	∞	

A comparison of the Schrödinger wave function with its approximation (32a) is carried out in Table II. The factor C_{kl} equals 1.424 in this case and could certainly not be neglected.

To get our equation for k we must fit (32) to (14) at the cut-off. The result is

$$\tan k\pi=-C_{kl}\frac{(k+l)!}{(k-l-1)!k^{2l+1}}\cdot\frac{J_{2l}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)-J_{2l+1}(x_{kl})J_{l-\frac{1}{2}}(\frac{1}{2}x_k^*)\dfrac{x_k^*}{x^*_{kl}}+\psi J_{2l+1}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)}{N_{2l}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)-N_{2l+1}(x_k)J_{l-\frac{1}{2}}(\frac{1}{2}x_k^*)\dfrac{x_k^*}{x^*_{kl}}+\psi N_{2l+1}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)}.\qquad(33a)$$

The factor ψ equals

$$\psi=\frac{2l+1}{x^*_{kl}}-\frac{2l+1}{x_{kl}}+\frac{1}{2\left(1-\dfrac{4l(l+1)}{x^2_{kl}}\right)}\times\left(\frac{1}{x_{kl}}+\frac{x^2}{x^{*3}_{kl}}-2\frac{x_k^{*2}}{x^{*3}_{kl}}\right).\qquad(34)$$

In spite of appearances to the contrary, the factor ψ tends to zero for small x and remains finite for $x_k=2[l(l+1)]^{\frac{1}{2}}$. The variables entering into (33a) and (34) are defined in (3), (25), (28), (29), (30), and (31).

It is an interesting accident that the factor entering (33a), namely,

$$C_{kl}\frac{(k+l)!}{(k-l-1)!k^{2l+1}},$$

is almost 1 for all possible values of k and l.

For instance, for $k=2$, $l=1$, we find 1.011, for $k=3$, $l=1$, we find 1.002 and so on. It reaches its largest value for $k=\infty$, $l=k-1$, namely, 1.074, while for $l\leqq k-2$, it is smaller than 1.026. Thus we may often neglect it entirely, and write for F, apart from a constant factor

$$F=x x_{kl}^{\frac{1}{2}} x_{kl}^{*-\frac{1}{2}}\{J_{2l+1}(x_{kl})\cos k\pi + N_{2l+1}(x_{kl})\sin k\pi\} \quad (32b)$$

and instead of (33a)

$$\tan k\pi=-\frac{J_{2l}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)-J_{2l+1}(x_{kl})J_{l-\frac{1}{2}}(\frac{1}{2}x_k^*)\dfrac{x_k^*}{x_{kl}^*}+\psi J_{2l+1}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)}{N_{2l}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)-N_{2l+1}(x_{kl})J_{l-\frac{1}{2}}(\frac{1}{2}x_k^*)\dfrac{x_k^*}{x_{kl}^*}+\psi N_{2l+1}(x_{kl})J_{l+\frac{1}{2}}(\frac{1}{2}x_k^*)}. \quad (33b)$$

If we apply Formula (33) to the $2p$ state studied by Tibbs[2] we find

$$k=2.004.$$

The approximate formula (21) gives

$$k=2.005.$$

Unfortunately a check with his result does not mean very much here because his value may lie anywhere between 2.00 and 2.041.

It seems to the author that the elimination of purely numerical work in quantum mechanics is very desirable because of the greater generality gained. The present paper makes the cut-off Coulomb field an analytically soluble case except for energy levels which are so close to the cut-off in the potential energy that only an exponential tail is situated beyond the cut-off radius. More precise criteria, together with formulas to replace (21), (27), and (33) will be given in another paper.

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Theory of Static Fields

I. A Phenomenological Attempt to Determine the Proper Field of an Electron

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According to the phenomenological theory of quantum fields, fluctuation phenomena correspond even to the lowest energy state of a system. The average value of field quantities is zero in the case of a vacuum. This value changes, however, if we introduce a particle into the field. The determination of the average value of the field fluctuation leads immediately to the expressions of the proper field of an electron. The calculation becomes considerably simplified by an appropriate use of Dirac's theory.

A. INTRODUCTION

1. Purpose of the Paper

IT is well known that classical theory does not represent, so far, a consistent scheme of physical phenomena, even if we apply it to a simple problem as the problem of the motion of an electron. Even the value of kinetic energy of an electron cannot be obtained from classical physics in a satisfactory way since the value given by the mechanical picture does not agree at all with the value one would expect from the transformation properties of static electromagnetic fields.

The first part of this paper intends to study the proper field of the electron from the point of view of quantum theory, in order to see whether or not quantum theory leads to modifications of the transformation properties of microscopic static fields which are likely to attenuate the difficulties met in classical physics. It answers at the same time the question in which way quantum theory accounts for static fields which are responsible for elementary macroscopic phenomena as electro- and magnetostatics.

The paper will give an account of a number of investigations which have been carried out during the last years by the author and a few of his collaborators and which could not yet be published in detail because of the events in Europe.

2. A Simple Example

In order to have a general view of the properties of the formalism of quantum field theory, we consider first a scalar field, described by a complete set of orthogonal functions in an arbitrary large cube of volume L^3.

$$\varphi_k = e^{i\omega_k t} u_k; \quad u_k = e^{i\mathbf{k}\mathbf{r}}/L^{\frac{3}{2}}. \tag{2.1}$$

These obey the Klein-Gordon equation

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \kappa^2\varphi = 0. \tag{2.2}$$

The static solution of Eq. (2.2) is given by

$$\varphi = -Ae^{-\kappa r}/r. \tag{2.3}$$

We shall see how quantum field theory accounts for this solution.

According to the formalism of quantum field theory, we have to consider every single Fourier component (2.1) as an oscillator, the energy of which can take only integral multiple values of $h\omega_k/2\pi$. If no particle is present in the field, the Hamiltonian of the field is given by

$$H = \sum_k \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2 - h\omega_k/2\pi), \tag{2.4}$$

where p_k and q_k are canonical variables,

$$p_k q_k - q_k p_k = h/2\pi i. \tag{2.5}$$

Then, the field is represented by the operator

$$\varphi = \sum_k \tfrac{1}{2}(4\pi)^{\frac{1}{2}} c \left\{ \left(q_k + \frac{1}{i\omega_k} p_k \right) u_k + \left(q_k - \frac{1}{i\omega_k} p_k \right) u_k^* \right\}. \tag{2.6}$$

It is well known that the Hamiltonian (2.4) describes fluctuating variables p_k and q_k, the average values of which are equal to zero. The average value of the field (2.6) in free space is, therefore, equal to zero.

Introducing an infinitely heavy particle with the coordinates $\mathbf{R}$ into the field, we can consider the kinetic energy of the particle to be a constant and the Hamiltonian of the system can be written

$$\begin{aligned} H = \sum_k \Big\{ & \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2 - h\omega_k/2\pi) \\ & + \tfrac{1}{2}(4\pi)^{\frac{1}{2}} c \cdot A \cdot \left[\left(q_k + \frac{1}{i\omega_k} p_k \right) \cdot u_k(R) \right. \\ & \left. + \left(q_k - \frac{1}{i\omega_k} p_k \right) \cdot u_k^*(R) \right] \Big\} \\ = \sum_k \tfrac{1}{2} \{ & (p_k + p_k^0)^2 + \omega_k^2 (q_k + q_k^0)^2 \\ & - h\omega_k/2\pi - p_k^{02} - \omega_k^2 q_k^{02} \}. \end{aligned} \tag{2.7}$$

The constant A plays the role of an interaction constant or "charge" in this theory. The abbreviations p_k^0 and q_k^0 are determined by

$$\begin{aligned} p_k^0 &= (4\pi)^{\frac{1}{2}} c \cdot A \cdot \frac{u_k(\mathbf{R}) - u_k^*(\mathbf{R})}{2i\omega_k}; \\ q_k^0 &= (4\pi)^{\frac{1}{2}} c \cdot A \cdot \frac{u_k(\mathbf{R}) + u_k^*(\mathbf{R})}{2\omega_k^2}. \end{aligned} \tag{2.8}$$

The Hamiltonian (2.7) represents an oscillator system, the average values of its canonical variables being shifted by the (negative) amounts given by (2.8). Introducing these average values in the expression (2.6), we obtain immediately the average value of the field

$$\bar{\varphi} = -2\pi c^2 A \sum_k \left\{ \frac{u_k^*(\mathbf{R}) u_k}{\omega_k^2} + \frac{u_k(\mathbf{R}) u_k^*}{\omega_k^2} \right\} = -A\frac{e^{-\kappa|\mathbf{r}-\mathbf{R}|}}{|\mathbf{r}-\mathbf{R}|} \tag{2.9}$$

in agreement with expression (2.3), since

$$(\omega_k/c)^2=k^2+\kappa^2$$

and

$$\sum_{\mathbf{k}}\frac{4\pi}{k^2+\kappa^2}\frac{e_\pm^{i\mathbf{k}(\mathbf{r}-\mathbf{R})}}{L^3}=\frac{e^{-\kappa|\mathbf{r}-\mathbf{R}|}}{|\mathbf{r}-\mathbf{R}|}.$$

The two last terms in (2.7) represent the infinite self-energy of our field, which, however, is without any interest for our purpose.

Replacing our simplified field model by the oscillator system representing the electromagnetic radiation field and the particle by a Dirac electron, the present scheme permits immediate calculation of the proper field of the electron to the approximation in which the electron is sufficiently heavy not to be appreciably influenced by the field.

Before, however, entering into the discussion of the proper field of the electron, we shall develop a mathematical apparatus, which will not only be helpful in the following calculations, but also simplify the considerations which we will have to present in the later parts of this series of investigations.

B. MATHEMATICAL APPARATUS

3. The Solution of Dirac's Equation

We shall write Dirac's equation in the following form:

$$-\frac{1}{c}\frac{\partial\Psi}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}\,\Psi+\frac{i\beta}{\Lambda}\Psi=0. \tag{3.1}$$

The constant Λ can be expressed by

$$\Lambda=h/2\pi mc, \tag{3.2}$$

the matrices $\boldsymbol{\alpha}$ and β are supposed to have the usual form.

We shall call a solution of Eq. (3.1) a complete set of orthogonal functions, depending on the variables x, y, z, t, $\boldsymbol{\alpha}$, and β. One verifies easily, that

$$\Psi_{\mathbf{p}}=\frac{p_0+mc-\boldsymbol{\gamma}\mathbf{p}}{\{L^3 2p_0(p_0+mc)\}^{\frac{1}{2}}}\cdot\exp\left[\frac{2\pi i}{h}(\beta p_0 ct+\mathbf{pr})\right] \tag{3.3}$$

is a complete solution of Eq. (3.1) if $\mathbf{p}$ denotes an arbitrary vector in space and

$$p_0^2-p^2=m^2c^2, \tag{3.4}$$

$$\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}. \tag{3.5}$$

We shall refer to the form (3.3) of the solution as the *standard form*.

In Table I the solution (3.3) is written explicitly, without the common normalizing factor.

Every column of (3.3) represents a set of four spinors which is generally called a partial solution of Dirac's equation. The four partial solutions included in (3.3) correspond to two sets of plain waves moving in opposite directions. This is an essential feature of the standard solution.

While the asterisk (*) denotes in the usual way the conjugate complex of a quantity, we shall denote by the sign "til" (˜) the conjugate transposed of a matrix. The conjugate transposed of Eq. (3.1) is

$$-\frac{1}{c}\frac{\partial\tilde{\Psi}}{\partial t}+\mathrm{grad}\,\tilde{\Psi}\boldsymbol{\alpha}-\frac{i}{\Lambda}\tilde{\Psi}\beta=0, \tag{3.6}$$

$$\tilde{\Psi}_{\mathbf{p}}=\exp\left[-\frac{2\pi i}{h}(\beta p_0 ct+\mathbf{pr})\right]\times\frac{p_0+mc+\boldsymbol{\gamma}\mathbf{p}}{\{L^3 2p_0(p_0+mc)\}^{\frac{1}{2}}}. \tag{3.7}$$

One verifies easily the normalization conditions

$$\int\tilde{\Psi}_{\mathbf{p}'}\Psi_{\mathbf{p}}d\tau=\delta_{\mathbf{p}'\mathbf{p}}, \tag{3.8}$$

$$\tilde{\Psi}_{\mathbf{p}}\Psi_{\mathbf{p}}=\frac{1}{L^3}, \tag{3.9}$$

$$\Psi_{\mathbf{p}}\tilde{\Psi}_{\mathbf{p}}=\frac{1}{L^3}. \tag{3.10}$$

Dirac's matrices describe quadrivalent variables, the four values of which we shall consider to belong to different values of the *character* of the electron. The character includes two bivalent sets of values, $(\alpha\delta)$ and $(\beta\gamma)$, belonging to different *spins*, and two bivalent sets, $(\alpha\beta)$ and $(\gamma\delta)$, belonging to different *signs* of the electron.

An electromagnetic field splits, for an observer in a given system of reference, into two parts: the electric and the magnetic field. In a some-

TABLE I. Solution of Eq. (3.3).

	$\exp\left[\frac{2\pi i}{h}(\mathbf{pr}-p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathbf{pr}-p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathbf{pr}+p_0ct)\right]$	$\exp\left[\frac{2\pi i}{h}(\mathbf{pr}+p_0ct)\right]$
α	p_0+mc	0	$+p_z$	$+(p_x-ip_y)$
β	0	p_0+mc	$+(p_x+ip_y)$	$-p_z$
γ	$-p_z$	$-(p_x-ip_y)$	p_0+mc	0
δ	$-(p_x+ip_y)$	$+p_z$	0	p_0+mc
	α	β	γ	δ

what analogous way, the character of an electron splits into spin and sign.

The relation (3.8) accounts for the orthogonality *in space*, the relation (3.9) expresses the orthogonality *in character*.

4. Dirac's Matrices

It is well known that Dirac's matrices $1, \beta, \alpha$ form a *fundamental system* of matrices. They permit the establishment of a complete *linear basis system* of 16 matrices which permits the expression of any matrix of four rows and columns by a linear expression. We shall denote the linear basis system as follows

$$\begin{array}{cc} & \beta \\ 1 & -\alpha \\ i\gamma & -\beta\sigma \\ \tau & -\sigma \\ & i\beta\tau \end{array} \tag{4.1}$$

with the following notations

$$\gamma=\beta\alpha; \quad \sigma=-\frac{i}{2}(\alpha\times\alpha); \quad \tau=\tfrac{1}{3}(\alpha\cdot\sigma). \tag{4.2}$$

The matrices (4.1) are of the Hermitian type.

We shall call matrices of the type

respectively *matrices of first* and *second kind*. The matrices

$$1, \quad \beta, \quad \sigma, \quad \beta\sigma$$

are of first kind;

$$\alpha, \quad i\gamma, \quad \tau, \quad i\beta\tau$$

of second kind.

In general, a matrix will contain parts of both kinds. We shall denote by

$$\begin{aligned} {}^1M&=\tfrac{1}{2}(M+\beta M\beta), \\ {}^2M&=\tfrac{1}{2}(M-\beta M\beta) \end{aligned} \tag{4.3}$$

the part of first and of second kind of the matrix M.[1]

5. The Cartan Transformation[2]

We introduce in Dirac's equation a general interaction term of Hermitian type, $U=\bar{U}$

$$-\frac{1}{c}\frac{\partial\Psi}{\partial t}+\alpha\,\mathrm{grad}\,\Psi+\frac{i}{\Lambda}\beta\Psi+\frac{2\pi i}{h}U\Psi=0. \tag{5.1}$$

The conjugate complex equation to (5.1) is

$$-\frac{1}{c}\frac{\partial\psi^*}{\partial t}+\alpha^*\,\mathrm{grad}\,\psi^*-\frac{i}{\Lambda}\beta\Psi^*-\frac{2\pi i}{h}U^*\Psi^*=0. \tag{5.2}$$

Introducing Cartan's matrix

$$\rho=-i\beta\alpha_y=\begin{pmatrix} & & +1 \\ & -1 & \\ +1 & & \end{pmatrix}, \quad \rho^2=1, \tag{5.3}$$

multiplying (5.2) from both sides by ρ and putting

$$\Phi=\rho\Psi^*\rho, \tag{5.4}$$

we obtain, because

$$\rho\beta\rho=-\beta, \quad \rho\alpha^*\rho=\alpha,$$

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[1] A more detailed investigation on the properties of Dirac's matrices will be found in Fernandes de Sa's thesis, Pôrto 1943, in which the general properties of the microscopic space-time continuum are discussed. We shall have to refer frequently to Fernandes de Sa's paper.

[2] The author is indebted to Mr. Jean Pirenne, Lyon, for having drawn his attention to this important transformation, given by E. Cartan.

the equation

$$-\frac{1}{c}\frac{\partial\Phi}{\partial t}+\alpha\,\mathrm{grad}\,\Phi+\frac{i}{\Lambda}\beta\Phi-\frac{2\pi i}{h}\rho U^*\rho\Phi=0. \quad (5.5)$$

In the general case, we have to put

$$\begin{aligned} U&=V+W,\\ V&=A_0+\alpha\mathbf{A}+i\gamma\mathbf{E}-\beta\sigma\mathbf{H}, \quad (5.6)\\ W&=\beta B+\tau C_0-\sigma\mathbf{C}+i\beta\tau D, \end{aligned}$$

where the A, B, C, D, E, and H's are real quantities, forming appropriately chosen tensors of a Lorentz transformation. Since

$$\rho V^*\rho=V;\quad \rho W^*\rho=-W$$

Eq. (5.5) takes the final form

$$-\frac{1}{c}\frac{\partial\Phi}{\partial t}+\alpha\,\mathrm{grad}\,\Phi+\frac{i}{\Lambda}\beta\Phi-\frac{2\pi i}{h}(V-W)\Phi=0. \quad (5.7)$$

Cartan's transformation provides us with a convenient means to study, with the help of Eq. (5.7) the behavior of states, which, according to the usually considered Eq. (5.1), belong to negative energy values. In the restricted case, in which only an electromagnetic field, represented by the four vector A_0, $\mathbf{A}$ is considered, we have to postulate that no additional assumption be introduced, which destroys the necessary symmetry corresponding to Cartan's transformation (Dirac's "filling up rule").

6. Spatial Rotation and Lorentz Transformation

Since we will not have to use these transformations explicitly, only a brief survey shall be given here. The general theory is contained in Fernandes de Sa's quoted paper.

A spatial rotation can be represented in this formalism by a left-hand operation

$$\Psi'=R\Psi. \quad (6.1)$$

The transformation matrix R is of first kind; it depends on Eulers angles φ, ϑ, ψ and on the matrices 1 and σ, $R(\varphi,\vartheta,\psi;1,\sigma)$.

An essential new feature appears as soon as we consider Lorentz transformations,

$$\Psi'=L_s\Psi, \quad (6.2)$$

where

$$L_s=Ch\frac{u}{2}-\frac{\mathbf{v}}{v}\alpha Sh\frac{u}{2}, \quad (6.3)$$

$$Ch\frac{u}{2}=\frac{1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}{\left\{2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\left[1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}};$$

$$Sh\frac{u}{2}=\frac{v/c}{\left\{2\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\left[1+\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}},$$

represents an ordinary spinor transformation.

We have to note that the spinor transformation (6.2) destroys the standard form of our solution. We have, therefore, after a spinor transformation, to rearrange our solution by a unitary transformation in order to reestablish the standard form. Physically, this fact means that the transformation properties of a fluctuating quantity of second kind does not obey the ordinary transformation laws under a Lorentz transformation.

7. General Motion

We shall call a general motion a complete set of orthogonal functions

$$\Phi_i=\Psi_{\mathbf{p}}s_{\mathbf{p}i};\quad s_{\mathbf{p}i}(t)=\int\bar{\Psi}_{\mathbf{p}}\Phi_i d\tau. \quad (7.1)$$

The transformation coefficients are matrices of four rows and columns, and depend explicitly on time and form a unitary transformation. If we decide to carry out summation on double indices without mentioning explicitly the sign of summation, they satisfy the conditions

$$\bar{s}_{i\mathbf{p}}\cdot s_{\mathbf{p}k}=\delta_{ik};\quad s_{\mathbf{p}'i}\cdot\bar{s}_{i\mathbf{p}}=\delta_{\mathbf{p}'\mathbf{p}}. \quad (7.2)$$

While in (7.1) $\Psi_{\mathbf{p}}$ represents a solution of Dirac's Eq. (3.1), Φ_i does not satisfy this equation. In general, it will be possible, however, to find a function U depending explicitly on time

in order to have Φ satisfying an equation of the type of (5.1).

C. THE PROPER FIELD OF AN ELECTRON

8. The Radiation Field

In order to account for the existence of photons, which according to the phenomenological theory has not to be explained but has simply to be taken as a postulate, we introduce the operators

$$\theta_{\mathbf{k}}, \quad N_{\mathbf{k}}=\frac{1}{i}\frac{\partial}{\partial\theta_{\mathbf{k}}} \tag{8.1}$$

operating on eigenfunctions of the type

$$\Gamma=\exp\left[i\sum_{k}\left(n_{\mathbf{k}}'\theta_{\mathbf{k}}+\frac{i}{2}\ln 2\pi\right)\right], \tag{8.2}$$

where the $n_{\mathbf{k}}'$ can have any positive or negative integer values including zero. If we want to interpret the numbers $n_{\mathbf{k}}'$ as numbers of photons, we have to exclude all solutions (8.2) in which all the values of $n_{\mathbf{k}}'$ do not have the same sign. We shall denote by $n_{\mathbf{k}}$ positive integer values including zero.

If we denote by

$$w_k=h\omega_k/2\pi \tag{8.3}$$

the energy of a photon corresponding to a wave vector $\mathbf{k}$, the eigenvalue problem of the radiation field becomes

$$\left(+\frac{\partial}{\partial t}+\frac{2\pi i}{h}\sum_{\mathbf{k}} w_k N_{\mathbf{k}}\right)\Gamma=0. \tag{8.4}$$

We have to attribute to the field energy the same sign as to the energy of the particles we want the field to interact with. We have, therefore, to introduce a scalar operator ϵ which represents the sign of the electron charge, $\epsilon^2=1$. We shall suppose ϵ to commute with Dirac's matrices. It is, for our considerations, an essential fact that Dirac's theory does not contain any operator of the type required here. Phenomenological theory has, therefore, to introduce this operator *ad hoc*, in an unsatisfactory way.

The solution of (8.4) may now be written

$$\Gamma=\exp\left[i\sum_{k}\left\{\frac{2\pi}{h}\eta_k\omega_k\epsilon t-\epsilon\eta_k\theta_k+\frac{i}{2}\ln 2\pi\right\}\right]. \tag{8.5}$$

The vector potential of the field will be determined below. We note, however, that the radiation field is described in classical theory by a complete set of orthogonal functions

$$\mathbf{A}_{\mathbf{k}}=\frac{hc}{(\pi\omega_k)^{\frac{1}{2}}}\mathbf{A}_{\mathbf{k}}u_{\mathbf{k}}, \tag{8.6}$$

where $\mathbf{A}_{\mathbf{k}}$ is a unit vector perpendicular to $\mathbf{k}$, determining the polarization of the plane wave. $u_{\mathbf{k}}$ has already been defined by (2.1). The amplitude in (8.6) is normalized to correspond to the energy of one photon w_k.

9. The Interaction Energy

For the interaction energy we can use the usual expression, but we need to take care that only electrons and photons of the same energy sign interact with each other. The interaction energy becomes

$$U(\epsilon)=\sum_{\mathbf{k}}-e\cdot\mathbf{A}_{\mathbf{k}}\cdot\boldsymbol{\alpha}\cdot\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\{\tfrac{1}{2}(1-\epsilon)\cdot[(N_{\mathbf{k}})^{\frac{1}{2}}\cdot\exp(i\theta_{\mathbf{k}})\cdot u_{\mathbf{k}}^{*}+\exp(-i\theta_{\mathbf{k}})\cdot(N_{\mathbf{k}})^{\frac{1}{2}}\cdot u_{\mathbf{k}}]$$
$$+\tfrac{1}{2}(1+\epsilon)\cdot[(N_{\mathbf{k}}^{*})^{\frac{1}{2}}\cdot\exp(-i\theta_{\mathbf{k}})\cdot u_{\mathbf{k}}+\exp(i\theta_{\mathbf{k}})\cdot(N_{\mathbf{k}}^{*})^{\frac{1}{2}}\cdot u_{\mathbf{k}}^{*}]\}. \tag{9.1}$$

The operators $\sqrt{N}$ and $\sqrt{N^*}$ can be expressed by a differentiation of the order $\frac{1}{2}$

$$\sqrt{N}=\frac{1}{\sqrt{i}}D^{\frac{1}{2}}; \quad \sqrt{N^*}=\frac{1}{\sqrt{-i}}D^{\frac{1}{2}}, \tag{9.2}$$

which is defined by

$$D^{\frac{1}{2}}e^{n\theta}=\sqrt{n}\cdot e^{n\theta}; \quad D^{\frac{1}{2}}\cdot D^{\frac{1}{2}}=\frac{\partial}{\partial\theta}.$$

With the help of (9.2) the interaction energy (9.1) can be written

$$U(\epsilon)=\sum_{\mathbf{k}}-e\cdot\mathbf{A}_{\mathbf{k}}\cdot\boldsymbol{\alpha}\cdot\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\cdot\frac{\exp\left(i\epsilon\frac{\pi}{4}\right)}{L^{\frac{3}{2}}}\{D_{\mathbf{k}}^{\frac{1}{2}}\exp\left[i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]+\exp\left[-i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]D_{\mathbf{k}}^{\frac{1}{2}}\}. \quad (9.3)$$

and we can consider

$$\mathbf{A}=\sum_{\mathbf{k}}\mathbf{A}_{\mathbf{k}}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\exp\left(i\epsilon\frac{\pi}{4}\right)}{L^{\frac{3}{2}}}\{D_{\mathbf{k}}^{\frac{1}{2}}\exp\left[i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]+\exp\left[-i\epsilon(\mathbf{kr}-\theta_{\mathbf{k}})\right]D_{\mathbf{k}}^{\frac{1}{2}}\} \quad (9.4)$$

to be the vector potential operator of the field.

The wave equation of the complete system becomes

$$\left\{-\frac{\partial}{\partial t}+c\boldsymbol{\alpha}\,\mathrm{grad}+\frac{i\beta}{\Lambda}+\frac{2\pi i}{h}U(\epsilon)-\frac{2\pi i}{h}\sum_{\mathbf{k}}w_k\cdot N_{\mathbf{k}}\right\}\bar{\Psi}=0 \quad (9.5)$$

and transforms under a Cartan transformation to

$$\left\{-\frac{\partial}{\partial t}+c\boldsymbol{\alpha}\,\mathrm{grad}+\frac{i\beta}{\Lambda}-\frac{2\pi i}{h}U(-\epsilon)-\frac{2\pi i}{h}\sum_{\mathbf{k}}w_k\cdot N_k\right\}\bar{\Phi}=0. \quad (9.6)$$

The functions $\bar{\Psi}$ and $\bar{\Phi}$ depend not only on the electron variables x, y, z, t, $\boldsymbol{\alpha}$, β, but also on the field variables ϵ, $\theta_{\mathbf{k}}$. The expressions (9.5) and (9.6) account correctly for the relation between electron and positron, while the usual interaction term, together with Dirac's filling up rule, does not account in a correct way for the proper field of an electron.

It can be objected, however, that our interaction term (9.3) is not in agreement with experiments since it does not contain any rule which permits the exclusion of spontaneous radiation transitions between states of different energy sign. This objection cannot be maintained. The quoted transitions occur only in second order while we shall show here that (9.3) applies only to non-relativistic first-order problems and that not only the interaction term, but even the whole scheme of quantum field theory has to be abandoned. The question of the quoted radiative transitions does, therefore, not appear explicitly in our considerations.

10. The Field Perturbation

We have now to solve, by an appropriate method of approximation, Eq. (9.5), and we must determine the average value of the ground state of the field (9.4). Supposing that no free photons are present in the field and that the electron has the momentum $\mathbf{p}$, we can represent the solution of (9.5) in first approximation by a series which corresponds to states in which one "virtual" photon has been emitted,

$$\bar{\Psi}=\Psi_{\mathbf{p}}\cdot\exp\left(-\sum\tfrac{1}{2}\ln 2\pi\right)+\sum_{\mathbf{p'k}}\Psi_{\mathbf{p'}}\cdot\exp\left(\frac{2\pi i}{h}\epsilon\omega_k t-i\epsilon\theta_k-\sum\tfrac{1}{2}\ln 2\pi\right)c_{\mathbf{p'k}}, \quad (10.1)$$

where the coefficient matrices $c_{\mathbf{p'k}}$ remain to be determined.

Introducing (10.1) into (9.5) we obtain immediately

$$\sum_{\mathbf{p'k}}\Psi_{\mathbf{p'}}\exp\left(\frac{2\pi i}{h}\epsilon\omega_k t-i\epsilon\theta_k-\sum\tfrac{1}{2}\ln 2\pi\right)\frac{dc_{\mathbf{p'k}}}{dt}=\frac{2\pi i}{h}U(\epsilon)\Psi_{\mathbf{p}}\exp\left(-\sum\tfrac{1}{2}\ln 2\pi\right)$$

and, with the help of relation (3.8)

$$\frac{dc_{\mathbf{p'k}}}{dt}=\frac{2\pi i}{h}\frac{1}{2\pi}\int\exp\left(i\epsilon\theta_k-\frac{2\pi i}{h}\epsilon\omega_k t\right)\times\bar{\psi}_{\mathbf{p'}}U(\epsilon)\psi_{\mathbf{p}}d\tau d\theta_{\mathbf{k}}. \quad (10.2)$$

In order to evaluate the relation (10.2), we remember that we are interested only in small values of momentum

$$\mathbf{p},\quad \mathbf{p}',\quad \frac{h}{2\pi}\mathbf{k}\ll mc. \tag{10.3}$$

The restriction (10.3) is due to the fact that we cannot expect our method to give reliable results as soon as the particle cannot be considered to be very heavy. We shall see later that our restriction limits our calculation of the proper field of the electron to large distances of the particle, $r\gg\Lambda$. For the present calculation, it implies, however, a considerable simplification, since we can put

$$p_0=p_0'=mc$$

and since we can neglect all terms which are of higher than first order in the momenta indicated in (10.3). One verifies easily that

$$\begin{aligned}\frac{dc_{\mathbf{p}'\mathbf{k}}}{dt}&=-\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{1}{2}}}\int\exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\tilde{\Psi}_{\mathbf{p}'}\boldsymbol{\alpha}\Psi_\mathbf{p}e^{i\epsilon\mathbf{k}\mathbf{r}}d\tau\\&=-\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{1}{2}}}\exp\left[-\frac{2\pi i}{h}(\beta mc^2+\epsilon\omega_k)t\right]\left\{\left(1+\frac{\boldsymbol{\gamma}\mathbf{p}'}{2mc}\right)\boldsymbol{\alpha}\left(1-\frac{\boldsymbol{\gamma}\mathbf{p}}{2mc}\right)\right\}\exp\left(\frac{2\pi i}{h}\beta mc^2t\right)\\&=-\frac{2\pi i}{h}\frac{ehc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{\mathbf{A}_\mathbf{k}}{L^{\frac{1}{2}}}\left\{\boldsymbol{\alpha}\exp\left(\frac{2\pi i}{h}2\beta mc^2t\right)-\frac{\beta}{2mc}[\mathbf{p}'+\mathbf{p}+i(\boldsymbol{\sigma}\times\mathbf{p}'-\mathbf{p})]\exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\right\},\end{aligned} \tag{10.4}$$

with

$$\mathbf{p}-\mathbf{p}'+\epsilon\frac{h}{2\pi}\mathbf{k}=0$$

and, finally,

$$c_{\mathbf{p}'\mathbf{k}}=-\frac{ehc}{(2\pi\omega_k L^3)^{\frac{1}{2}}}\left\{\frac{\mathbf{A}_\mathbf{k}\boldsymbol{\alpha}\cdot\boldsymbol{\beta}}{2mc^2}\exp\left[\frac{2\pi i}{h}(2\beta mc^2-\epsilon\omega_k)t\right]-\frac{\beta}{2mc\epsilon\omega_k}\mathbf{A}_\mathbf{k}\left[2\mathbf{p}+i\left(\boldsymbol{\sigma}\times\epsilon\frac{h}{2\pi}\mathbf{k}\right)\right]\exp\left(-\frac{2\pi i}{h}\epsilon\omega_k t\right)\right\}. \tag{10.5}$$

11. The Average Value of the Field

We now have to calculate the average value of the field (9.4),

$$\frac{1}{L^3}\bar{\mathbf{A}}=\int\bar{\bar{\Psi}}\mathbf{A}\bar{\Psi}d\theta. \tag{11.1}$$

We observe that the field $\mathbf{A}$ depends on coördinates $\mathbf{r}$ in space while the functions $\bar{\Psi}$ depend on the electron coordinates, which in this paragraph shall be denoted by capital letters $\mathbf{R}$. The average field depends, therefore, on time, on the coordinates $\mathbf{r}$ and $\mathbf{R}$, and on the variables ϵ, $\boldsymbol{\alpha}$, and β.

Introducing in (11.1) the respective values from (9.4) and (10.1), we find

$$\frac{1}{L^3}\bar{\mathbf{A}}=\sum_\mathbf{k}\mathbf{A}_\mathbf{k}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{1}{L^{\frac{1}{2}}}\left\{\bar{c}_{\mathbf{p}'\mathbf{k}}\tilde{\Psi}_{\mathbf{p}'}\exp\left(i\epsilon\mathbf{k}\mathbf{r}-\frac{2\pi i}{h}\epsilon\omega_k t\right)\psi_\mathbf{p}+\tilde{\Psi}_\mathbf{p}\exp\left(-i\epsilon\mathbf{k}\mathbf{r}+\frac{2\pi i}{h}\epsilon\omega_k t\right)\Psi_{\mathbf{p}'}c_{\mathbf{p}'\mathbf{k}}\right\}.$$

In our approximation, the functions $\bar{\Psi}$ can be replaced, a part from a factor $L^{-\frac{1}{2}}$, directly by the respective exponentials,

$$\bar{\mathbf{A}}=\sum_\mathbf{k}\mathbf{A}_\mathbf{k}\frac{hc}{(2\pi\omega_k)^{\frac{1}{2}}}\frac{1}{L^{\frac{1}{2}}}\left\{\bar{c}_{\mathbf{p}'\mathbf{k}}\exp\left[-\frac{2\pi i}{h}\epsilon\omega_k t+\frac{2\pi i}{h}(\mathbf{p}-\mathbf{p}')\mathbf{R}+i\epsilon\mathbf{k}\mathbf{r}\right]+\exp\left[\frac{2\pi i}{h}\epsilon\omega_k t-\frac{2\pi i}{h}(\mathbf{p}-\mathbf{p}')\mathbf{R}-i\epsilon\mathbf{k}\mathbf{r}\right]\cdot c_{\mathbf{p}'\mathbf{k}}\right\},$$

where we introduce the values of the coefficients $c_{\mathbf{p}'\mathbf{k}}$ found in (10.5). The result of the calculation is

$$\bar{\mathbf{A}}=\mathbf{A}_e+\mathbf{A}_\sigma+\mathbf{A}_\alpha \tag{11.2}$$

with

$$\mathbf{A}_e=\sum_{\mathbf{k}}+e\frac{h^2c^2}{2\pi\omega_k L^3}\frac{\mathbf{a}_k\mathbf{p}}{mc\epsilon\omega_k}\mathbf{a}_k\beta\{e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}+e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\}, \tag{11.3}$$

$$\mathbf{A}_\sigma=\sum_{\mathbf{k}}-e\frac{h^2c^2}{2\pi\omega_k L^3}\frac{\beta}{2mc\epsilon\omega_k}\mathbf{A}_k\left[\left(\sigma\times i\epsilon\frac{h}{2\pi}\mathbf{k}\right)\mathbf{A}_k\right]\{e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}-e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\}. \tag{11.4}$$

$$\mathbf{A}_\alpha=\sum_{\mathbf{k}}-e\frac{h^2c^2}{2\pi\omega_k L^3}\left\{\exp\left(-\frac{2\pi i}{h}2\beta mc^2t\right)\mathbf{A}_k\beta\frac{\mathbf{A}_k\alpha}{2mc^2}e^{i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}+\mathbf{A}_k\frac{\mathbf{A}_k\alpha}{2mc^2}\beta\exp\left(\frac{2\pi i}{h}2\beta mc^2t\right)e^{-i\epsilon\mathbf{k}(\mathbf{r}-\mathbf{R})}\right\}. \tag{11.5}$$

12. The Summation of the Series

The series (11.3), (11.4), and (11.5) can be evaluated. We note that the summation has to be carried out not only on the wave vector $\mathbf{k}$, but also on both directions of polarization. One verifies easily the following relations

$$\sum A_x^2=1-\frac{k_x^2}{k^2};\quad\cdots,\qquad \sum A_xA_y=-\frac{k_xk_y}{k^2};\quad\cdots \tag{12.1}$$

$$\frac{1}{r}=\sum_{\mathbf{k}}\frac{4\pi}{L^3}\frac{1}{k^2}e^{i\mathbf{k}\mathbf{r}},\qquad \frac{1}{r^2}=\sum_{\mathbf{k}}\frac{2\pi^2}{L^3}\frac{1}{k}e^{i\mathbf{k}\mathbf{r}},\qquad \operatorname{grad}\left(\frac{1}{r}\right)=\sum_{\mathbf{k}}\frac{4\pi}{L^3}\frac{i\mathbf{k}}{k^2}e^{i\mathbf{k}\mathbf{r}},\qquad \text{etc.} \tag{12.2}$$

With the help of (12.1) and (12.2), we find immediately

$$\mathbf{A}_e=\beta\epsilon e\left\{\frac{\mathbf{v}}{c}\frac{1}{|\mathbf{r}-\mathbf{R}|}-\tfrac{1}{2}\operatorname{grad}\frac{\mathbf{v}(\mathbf{r}-\mathbf{R})}{c|\mathbf{r}-\mathbf{R}|}\right\}, \tag{12.3}$$

$$\mathbf{A}_\sigma=\epsilon\mu\beta\frac{(\sigma\times\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^3}, \tag{12.4}$$

$$\mathbf{A}_\alpha=\frac{\mu}{\pi}\left\{\left(\frac{\alpha}{|\mathbf{r}-\mathbf{R}|^2}-\operatorname{grad}\frac{\alpha(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right)\beta+\beta\left(\frac{\alpha}{|\mathbf{r}-\mathbf{R}|^2}-\operatorname{grad}\frac{\alpha(\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\right)\right\}\exp\left(\frac{2\pi i}{h}2\beta mc^2t\right), \tag{12.5}$$

with

$$\frac{\mathbf{v}}{c}=\frac{\mathbf{p}}{mc},\qquad \mu=\frac{eh}{4\pi mc}.$$

13. The Proper Field of the Electron

We now have to interpret the expressions (12.3), (12.4), and (12.5). Since the velocity of an electron corresponding to our standard solution (3.3) is equal to $-\beta\mathbf{v}$, $\mathbf{A}_e$ corresponds directly to the magnetic field because of the translation of the Coulomb field

$$A_0=\epsilon\frac{e}{|\mathbf{r}-\mathbf{R}|}. \tag{13.1}$$

The gradient in the vector potential (12.3) does not contribute to the magnetic field and has no physical signification.

$$\mathbf{A}_\sigma=\epsilon\mu\beta\frac{(\sigma\times\mathbf{r}-\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^3} \tag{13.2}$$

represents the vector potential of the magnetic spin field of the electron.

The last field vanishes, since β anticommutes with α

$$\mathbf{A}_\alpha=0. \tag{13.3}$$

The proper field of the electron consists, therefore, in our approximation of two parts which belong to the charge and to the magnetic moment of the electron. It obeys Maxwell's equations and accounts correctly for electrostatic and magnetostatic phenomena.

The field $\mathbf{A}_\alpha$ is of second kind. It corresponds to excitations of the field due to virtual pair production. Its average value vanishes. Still, we know that this field plays an important part. Mr. Jean Pirenne has made a thorough investi-

gation about the way in which the field $\mathbf{A}_\alpha$ contributes to the interaction of electrons. He finds, in agreement with (13.3), that no contributions to interaction result at finite distances. Still, the field $\mathbf{A}_\alpha$ gives rise to interaction terms between electrons and positrons at very small distances. These interaction terms are of the order of spin-spin interaction and can be interpreted as forces of the Heisenberg and Majorana type between unlike particles.[3]

14. Thibaud's Experiment

The next task is to investigate the terms of higher order of the electron field. The higher order terms which have been calculated[4] are rather complicated and no longer obey Maxwell's equations. If we want to attribute a physical meaning to them, electrodynamics becomes an open scheme which does not longer permit any theoretical predictions. If the difficulties concerning the transformation properties of static fields are due to deviations of Maxwell's theory, the question arises whether or not Maxwell's theory represents a complete description of the electromagnetic field in vacuum.

Owing to Professor Jean Thibaud's kind interest in the problem, it has been possible to examine this question from the experimental point of view. Let us suppose that a small fraction of the electromagnetic field is not described by a vector potential but by an antisymmetric tensor potential, as some of our quoted calculations indicated. It can be easily shown that such a potential corresponds to the existence of longitudinal photons of very high penetrating power which could, however, be detected if their wave-length approaches Compton's wave-length. Professor Thibaud examined the radiation due to electron impacts under a voltage of about 280 kv. The penetrating power of the radiation has been studied with a thick layer of lead, up to 140 mm. The result was negative; no indications for longitudinal photons were found; the penetrating part of the radiation corresponded exactly to the known cosmic radiation.[5]

We concluded from Thibaud's results that the origin of the fundamental difficulties concerning the transformation properties of static fields is not the electromagnetic part of the problem and has to be searched for elsewhere.

15. Conclusions

The following conclusions can be drawn from the first part of this paper:

1. Elementary electrostatic facts lead to a consistent mathematical scheme which can be applied to Dirac's theory.
2. The phenomenological scheme of quantum field theory is not appropriate to our problem and has to be abandoned.
3. The proper field of the electron depends essentially on the character of the electron.
4. No progress of the theory is possible without a careful study of the kinematical behavior of the electron character according to Dirac's theory.

That study shall be communicated in the second part of this paper.

ACKNOWLEDGMENTS

The first part of this paper was completed in France. The author desires to express his respectful thanks to Professor Paul Langevin (Paris) for the possibility of working in Paris in 1938. His thanks are due to his friend Jacques Salomon (Paris). The author thanks respectfully Professor Louis de Broglie for the support given him. The author is very much indebted to Professor Jean Thibaud (Lyon) for his kind interest in this work, and it is a great pleasure to him to express his gratitude to his friend, Professor Max Morand (Lyon), who did not miss one single opportunity, official or private, to facilitate this work. The author desires to thank his young friends, Jean Pirenne (Lyon) and Fernandes de Sa (Porto) for the enthusiasm with which they assisted this work under, indeed, very difficult conditions. Last but not least, the author expresses his gratitude to Professor Enrique Gaviola (Córdoba) for the possibility of continuing this work at his observatory.

[3] Jean Pirenne, Thèse, Lyon (1941). The author has no possibility of knowing at present whether or not this paper has been published in the meantime. See, however, Cahiers de physique **4**, 1, 6 (1941).

[4] One part of these calculations has been made by Mr. S. Höxter.

[5] The author is indebted to Professor Jean Thibaud for the communication of his unpublished results by letter of November 7, 1940 and for his kind permission to refer to his measurements.

Letters to the Editor

PROMPT publication of brief reports of important discoveries in physics may be secured by addressing them to this department. The closing date for this department is the third of the month. Because of the late closing date for the section no proof can be shown to authors. The Board of Editors does not hold itself responsible for the opinions expressed by the correspondents. Communications should not in general exceed 600 words in length.

New Planetary Systems

H. K. SEN
Allahabad University, Allahabad, India
October 1, 1943

OBSERVATION has at last succeeded in giving us a verdict on the vexed problem which had been engaging the attention of the theorists for some time—whether our solar system is the only one of its kind in this vast universe. From the study of parallax observations Dr. K. Aa. Strand[1] has recently been able to prove the existence of a planetary body companion to the 61 Cygni system. The system shows a periodic deviation from the Keplerian orbit, and this can only be explained on the supposition that a third body (61 Cygni *C*) of small mass, about sixteen times that of Jupiter, revolves around one of the stars of the binary system. The luminosity of 61 Cygni *C* is found to be very low, and it can very well be classified as a planet.

A parallel discovery to 61 Cygni *C* has also been almost concurrently made. Reuyl and Holmberg[2] have discovered a similar planetary body of even smaller mass in the binary star system, 70 Ophiuchi. It is worthy of note that neither of the stars 61 Cygni and 70 Ophiuchi exceeds a distance of 10 parsecs from the sun.

We have, at last, evidence of the discovery of two planetary systems other than our own, and, what is more, these happen to be very near our own system. The suggestion naturally comes to mind that we may by similar means be able to find many more planetary systems in our galactic system, and the number of planetary systems in this universe may be quite large. It will be interesting in this connection to review briefly what theory has to say on this question.

On the tidal theories of Jeans[3] and Jeffreys,[4] planetary systems should indeed be rare. Taking the generally accepted age of the universe as of the order of 10^9 to 10^{10} years, and Jeans' estimate (1929) of the frequency of planetary systems on his theory—about one per 5000 million years—we have at the most two planetary systems in our galactic system. On Lyttleton's[5] binary star collision theory, the probability of the formation of planetary systems is almost nil.

The probability is much greater on Banerji's[6] Cepheid theory (1942) in which a passing star is supposed to break up an oscillating Cepheid. Encounter at only a moderate distance is necessary in this theory—it need not be grazing or close. Banerji has stated his conclusion in the following words, "One conclusion seems to be irresistible. If the theory be correct in its essentials, there may be more planetary systems than at present supposed." The observations thus definitely favor Banerji's conclusion.

It may be remarked that the presence of planetary systems in binary stars is very plausible on the Cepheid theory of the origin of binary stars recently advanced by the author.[7] A Cepheid is supposed to break up into a binary star system on account of increase of angular velocity due to radiation of energy. The filament connecting the two stars affords a chance for planets to be formed by condensation.

[1] K. Aa. Strand, Pub. Astronom. Soc. Pac. **55**, 29 (1943).
[2] Reuyl and Holmberg, Astrophys. J. **97**, 41 (1943).
[3] J. H. Jeans, *The Problems of Cosmogony and Stellar Dynamics* (1919), p. 275; *Astronomy and Cosmogony* (1929), pp. 7, 22, 409.
[4] H. Jeffreys, *The Earth* (1929), p. 16.
[5] R. A. Lyttleton, M.N.R.A.S. **98**, 536 (1938).
[6] A. C. Banerji, Proc. Nat. Inst. Sci. Ind. **8**, 173–197 (1942).
[7] H. K. Sen, Sci. and Cult. **7**, 582 (1941–1942).

Quantized Probability

ALFRED N. GOLDSMITH
580 Fifth Avenue, New York, New York
November 13, 1943

THIS communication proposes for development the theory that there exists a finite unit of probability, representing the lower limit of possibility. (A suggested designation for this unit is the "chance-quantum," abbreviated cq.) Thus, if the probability of an event is equal to or greater than one cq., it may ultimately occur. If its probability is less than one cq., it will never occur. The latter statement constitutes a break with classic probability theory.

Probability-quantization theory results from consideration of hypothetical experiments wherein, for example, a sufficiently great multiplicity of monkeys operating at random an adequately large number of typewriters will ultimately reproduce master-works of literature. Inherent aversion to this conclusion has led some to hold that the assumed result would necessarily be interpreted as fraudulently achieved. This mode of escape from the assumed conclusion can perhaps be avoided by postulating that the probability of thus reproducing great literature is so small as to fall below one cq. Accordingly the assumed result becomes flatly impossible.

If probability-quantization is an existent condition in nature, certain interesting theoretical consequences appear to follow. These include:

1. If an event has a calculated probability of less than one cq., it will not occur.

2. Assuming the validity of Bayes' theorem, events can occur only if both their existence probability and their likelihood equal or exceed one cq., and if their product exceeds one cq.

3. Events may differ in probability only by an integral number of cq.

4. The occurrence of events having a probability of one or relatively few cq. will be not only extremely infrequent but may have an unusual form of random distribution.

5. For an event having an appreciable probability (equivalent to many cq.), a change in surrounding conditions leading to a computed change in probability of less than one cq. will in fact cause no change in the probability of the event.

6. The interrelations between the cq. and the energy quantum (as well as the elsewhere proposed time quantum) may be close and significant.

An absolute proof of the existence of the cq. faces the usual difficulties of proving a universal negative. At best, the establishment of a high probability for the existence of the cq. is attainable. A possible experimental method might be based on the following thoughts. Suppose that there exist n possible but very improbable events of a similar and identifiable type. If the selected one of these events not only does not occur in n trials, but also does not occur in mn trials (where m is a large number), there is a likelihood that it will not occur at all and that its probability lies below the assumed unit cq.

Such an experiment might be attempted using, as a basis, mechanical, electrical, or electronic coincidences. Illustratively, there may be used a set of coaxially rotating disks, propelled pneumatically along their peripheries, mounted in low friction bearings, and mechanically isolated from each other. Each disk bears a circular perforation of the same radius, and at the same distance from the disk center. A collimated light beam, parallel to the rotational axis, can pass through the perforations when lined up, and thereafter be registered by conventional photoelectric means. Control of perforation size, perforation distance from the axis, and number of disks, will enable selection of desired probability parameters. If a coincidence of extremely small probability did not occur in a time which was a large multiple of the time of its probable occurrence, the hypothesis of probability-quantization would be strengthened. Great care might be necessary to avoid confusing quantum effects in the physical system with quantization of probability.

If the cq. theory is found valid, interesting consequences may follow. Philosophically, the universe will be a system subject to prediction (determinism) for events of probability less than one cq. and to non-prediction ("free will" or normal chance) for events of probability above one cq. Probability theory for ordinarily likely events will be little affected. But the probability of small-scale and extremely unlikely effects may be largely controlled by the new theory—a matter of considerable interest in certain modern microphysical realms. And the comparison between continuous probability and quantized mechanics, on the one hand, and quantized probability and continuous mechanics, on the other hand, may well merit study.

It is hoped that others will see fit to develop probability-quantization theory and its implications. The author particularly wishes to express his deep appreciation of the searching and constructive comments on this theory which, in the course of correspondence, he has received from Professor F. Bernstein, Dr. Thornton D. Fry, and Professors Harold Hotelling and Richard von Mises, as well as other interested men of science.

Proceedings of the American Physical Society

Minutes of the Meeting of the New York State Section Held at Ithaca, New York, November 6, 1943

THE Fall Meeting of the New York State Section of the American Physical Society was held at Baker Hall and Rockefeller Hall on Saturday, November 6, 1943. An attendance of over ninety members and visitors was recorded at the sessions.

The following program of invited papers was presented:

Morning Session

Address of Welcome. G. H. Sabine, *Vice President, Cornell University.*

Education of Physicists for Industry. J. T. Lyttleton, *Corning Glass Works.*

Metallic Mirrors by the Evaporation Process. A. Fogelsonger, *Evaporated Metal Film Corporation.*

Coulomb's Laws, What They Are and What They Are Not. C. C. Murdock, *Cornell University.*

Order and Entropy at the Absolute Zero. K. K. Darrow, *Bell Telephone Laboratories.*

Afternoon Session

Discussion: The Army and Navy Training Programs

Army Basic Course. H. E. Howe, *Cornell University.*

Navy Basic Course. P. R. Gleason, *Colgate University.*

Premeteorology Course. G. H. Cameron, *Hamilton College.*

Preinduction Training. P. G. Johnson, *Cornell University.*

W. R. Fredrickson
Secretary

Proceedings of the American Physical Society

Minutes of the Meeting of the Metropolitan Section Held at Hunter College, New York, October 23, 1943

THE first meeting of the Metropolitan Section of the American Physical Society for the season 1943–1944 was held on Saturday, October 23, 1943, at Hunter College, New York City. About seventy-five people attended. The programme was as follows:

Afternoon Session

Contributed Papers:

Interpretation of the Spinning Electron with Bipolar Coordinates. Mary Hewlett Payne, *Brown University.*

Absorption of Gamma-Radiation in Terrestrial Material. C. W. Sheppard, *Massachusetts Institute of Technology* (By title).

Invited Papers:

The Study of the Ionization of the Atmosphere by Cosmic and Local Radiation. V. F. Hess, *Fordham University.*

Production of Neutrons by Cosmic Radiation. S. A. Korff, *New York University.*

The Effect of Centrifugal Force on the Electromotive Force of Galvanic Cells. D. A. MacInnes, *Rockefeller Institute for Medical Research.*

The Field Emission Arc in its Application to High Speed Radiography. C. M. Slack, *Westinghouse Electric and Manufacturing Company.*

At the conclusion of the afternoon session there was an inspection of the Physics Laboratories of Hunter College. The inspection was followed by the dinner of the Metropolitan Section in the restaurant of Hunter College.

Evening Session

Invited Papers:

The New Home of the American Institute of Physics. H. A. BARTON, *American Institute of Physics.*

The Chemical Age of Plenty. F. C. WHITMORE, *Pennsylvania State College.*

W. S. GORTON
Secretary-Treasurer

Absorption of γ-Radiation in Terrestrial Materials. C. W. SHEPPARD, *Massachusetts Institute of Technology.*—As part of a research program on the role of radioactivity in petroleum genesis calculations have been made of the absorption of γ-rays in terrestrial materials. The intensity of ionization has been calculated for a steel ionization chamber having thick walls and containing air at one atmosphere. In the case where the chamber is situated above a semi-infinite solid absorber containing m_{Ra}, m_{Th}, and m_K grams of Ra, Th, and K per gram, respectively, the number of ion pairs per cc per second will be $1.4\times10^{12}m_{Ra}+2.4\times10^{8}m_{Th}+27m_K$. The expression was tested by inserting the measured Ra content[1] and Th content[2] of selected samples from the Millikan rock series, together with an estimate of the K content. The results were compared with Millikan's measured values using the factor (1/13.8) to reduce the readings to 1 atmosphere.[1] No systematic disagreement was found. Large individual deviations in both directions indicated sampling irregularities. The lower value of ionization estimated by Hess[3] may be attributed to under-estimation of the effect of scattered low energy quanta on the ionization chamber.

[1] R. D. Evans and R. W. Raitt, Phys. Rev. **48**, 171 (1935).
[2] R. W. Raitt, Ph.D. Thesis, California Institute of Technology (1935).
[3] V. F. Hess, Terr. Mag. **46**, 409 (1941).

Proceedings of the American Physical Society

MINUTES OF THE MEETING OF THE NEW ENGLAND SECTION HELD AT THE CONNECTICUT COLLEGE FOR WOMEN, NEW LONDON, CONNECTICUT, OCTOBER 16, 1943

THE twenty-second regular meeting of the New England Section of the American Physical Society was held at the Connecticut College for Women, New London, Connecticut, on Saturday, October 16, 1943. Forty-five members were in attendance. The invited papers were as follows:

Discussion on **"Problems Connected with the V-12 Program"** led by V. E. EATON, *Wesleyan University.*

Physics at Connecticut College. G. K. DAGHLIAN, *Connecticut College.*

Some Crude, Simple, Large Scale Models of Apparatus Developed for Teaching First-Year College Physics. J. BARTON HOAG, *U. S. Coast Guard Academy.*

Fast Neutron Absorption in Gases, Walls, and Tissue. GLADYS A. ANSLOW, *Smith College.*

Is Hardness a Physical Property of Solid Matter? S. R. WILLIAMS, *Amherst College.*

But one ten-minute paper was presented for which the abstract follows:

Application of Dirac's New Field Theory to the Calculation of Higher Order Matrix Elements. SHUICHI KUSAKA, *Smith College.*—Dirac's new method of eliminating the divergences in field theories is applied to the calculation of higher order matrix elements which diverge in the usual theory. For an electromagnetic field interacting with charged particles, all these matrix elements vanish, and hence this theory coincides with the theory of radiation damping developed by Heitler, Wilson, and others in which the matrix elements which diverge in the usual theory are neglected. For all types of meson fields interacting with nucleons, however, Dirac's theory gives finite but non-vanishing values for the higher order matrix elements, and hence does not agree with the current theory of radiation damping. Moreover, these matrix elements are of the same order of magnitude as the lowest order matrix element even for non-relativistic energies, and become much larger for higher energies. Thus the perturbation theory is inadequate for the treatment of these problems.

At the business meeting a resolution was presented to the meeting in commemoration of Arthur P. R. Wadlund of Trinity College, who was Vice Chairman of the Section at the time of his lamented death. The following officers were elected for the ensuing year:

Chairman: Gladys A. Anslow,
Vice Chairman: Morton Masius,
Secretary-Treasurer: Mildred Allen,
Members of Programme Committee: Clarence E. Bennett, W. W. Stifler.

MILDRED ALLEN, *Secretary-Treasurer*

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